MATHEMATICS

CLASS XI

(Under AHSEC Curriculum)

Content Developed by-

Dr. Dipak Sarma
&
Dr. Parag Kumar Deb
Associate Professor
Dept. of Mathematics
Cotton College, Assam

Dr. Anjana Bhattacharyya
Associate Professor
Dept. of Mathematics
B. Borooah. College, Assam

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Mathematical Reasoning (Chapter 14)

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Mathematical Reasoning Chapter 14

Statements (or Logical Statements):

In our daily life we all use sentences which are distinguished as declarative, imperative, interrogative and exclamatory. But by a mathematical statement (or simply statement or proposition) we mean a sentence which is declarative (or assertive) and about which it is possible to say if it is true or false. Thus all sentences are not statements.

The following sentences are statements (or Mathematical statements):

(i) The weight of an elephant is less than that of man.
(ii) 5 is greater than 4.
(iii) The sum of two sides of a triangle is greater than the third side.
(iv) $a^2 - b^2 = (a-b)(a+b)$ for all values of a and b.

We see that sentence is false and the other three sentences are all true. All the above sentences are declarative and for each of these it is possible to say whether it is true or false. Hence all the four sentences are statements.

Now consider the sentence 'Girls are more intelligent than boys'.

In this case, we cannot say whether the sentence is always true or false. Because some people may think it is true and others may not agree i.e., the sentence is vague or confusing and it is not acceptable as a statement in Mathematics.

None of the following is a statement:

1. How nice ! 2. This is true. 3. What do mean? 4. May God bless you.

The sentences 1, 2, 3 and 4 are respectively exclamatory, imperative, interrogative and imperative. These sentences are not declarative and this is not possible to say whether anyone of them is true or false. Hence none of these sentences is a statement.

A sentence is called a mathematically acceptable statement or Mathematical statement if it is either true or false but not both. A statement is denoted by the letter p or q or r.

Illustration 1: *Consider the sentences given below:*

(i) Delhi is the capital of India. (ii) Earth revolves around the Sun.

(iii) Two plus one is four. (iv) Every rhombus is a parallelogram.

(v) All prime numbers are odd numbers.

Each of the above declarative sentences is a statement. Of these statements (i), (ii), (iv) are all true, but (iii) and (v) are false.

Illustration 2: *Consider the sentences given below:*

(i) Give me a pen. (ii) Go to the market. (iii) Put off the fan.

(iv) Complete your work today. (v) Please help me.

All the above sentences are imperative sentences which express either a request or an order and none of them has a truth value. Hence none of them is a statement.

Illustration 3: *Consider the sentences given below:*

(i) How is your mother? (ii) What are you doing?

(iii) Where are you going? (iv) Have you solved the problem?

(v) Is every triangle equilateral?

Clearly, each sentence is a question i.e., *interrogative*. None of the above sentence is a statement, since none of them has a truth value.

Illustration 4: *Consider the sentences given below:*

(i) How nice ! (ii) How beautiful ! (iii) How fine !

(iv) How tall is the monument ! (v) How big is the elephant !

All the above sentences are *exclamatory*. None of them has any truth value and hence none of them is a statement.

Illustration 5: *Consider the sentences given below:*

(i) May God bless you ! (ii) May long live the king !

These sentences have no truth values and hence they are not statements. These are exclamatory sentences.

Remarks: Sentences involving the terms 'today' or 'yesterday' are not statements. Because it is not known, what time is mentioned in such cases. For example, the sentences 'Today is Sunday', 'Tomorrow is Tuesday', 'Yesterday was Friday' are not statements. Because these sentences have no truth values.

Truth value of a Statement:

The truth or falsity of a statement is called its *truth value*. Truth of a statement is denoted by T and falsity of a statement is denoted by F.

Example 1. *Which of the following sentences are statements? Give reasons for your answer.*

(i) There are 35 days in a month.
(ii) Mathematics is difficult.
(iii) The sum of 5 and 7 is greater than 10.
(iv) The square of a number is an even 10 number.
(v) The sides of a quadrilateral have equal lengths.

(vi) Answer their question.
(vii) The product of (-1) and 8 is 8.
(viii) The sum of interior angles of a triangle is 180^0.
(ix) Today is a windy day.
(x) All real numbers are complex numbers.

Solution:

(i) Statement: It is false to say that month has 35 days.
(ii) Sentence: Mathematics may be difficult for one, but may be easy for the other.
(iii) Statement: It is true that sum of 5 and 7 is greater than 10.
(iv) Sentence: The square of a number may be even or it may be odd, e.g. $2^2 = 4$ (even), $3^2 = 9$ (odd).
(v) Sentence: A quadrilateral may have equal sides as it may be a rhombus or a square or the quadrilateral may have unequal sides.
(vi) Sentence: It is an order.
(vii) Statement: It is false because product of (-1) and 8 is not 8.
(viii) Statement: It is true that sum of interior angles of a triangle is 180^0.
(ix) Sentence: It is a windy day. Which day it is?
(x) Statement: It is true that all real numbers are complex numbers of the form a + i.0.

Example 2. *Give three examples of sentences which are not statements. Give reasons for the answer.*

Solution: (i) It is a rainy month. Which month is a rainy month? We cannot say which month is rainy.

(ii) She is very beautiful. Who is beautiful?
(iii) You are a brave boy. Who is brave?

Example 3. *Examine with reasons whether the following sentences are statements:*

(i) $\sqrt{5}$ is an irrational number. (ii) How far is Delhi from here?

(iii) $x^2 - 5x + 6 = 0$. (iv) There is no rain without clouds.

Solution: (i) $\sqrt{5}$ is an irrational number is true i.e. the sentence has a truth value 'true'. Hence it is a statement.

(ii) It is an interrogative sentence which also contains the vague word 'here'. Hence it is not a statement.
(iii) As the values of x are not known, $x^2 - 5x + 6 = 0$ has no definite truth value. Hence $x^2 - 5x + 6 = 0$ is not a statement.
(iv) It is established by scientists that cloud is formed before it rains. Thus the sentence is always true. Hence it is a statement.

Exercise 14.1

1. Which of the following are logical statements and which are not. Indicate the truth values of the statements:

(i) There are only finite number of rational numbers.

(ii) Congruent triangles are similar.

(iii) $3+5>4+2$.

(iv) Two non-empty sets have always a non-empty intersection.

(v) $x+7=23$.

(vi) Put the home work on the black-board.

(vii) All differentiable functions are continuous.

(viii) Snakes can fly.

(ix) He stood first in the class.

(x) $\sin x>1$ for every x, x is measured in radians.

2. Which of the following sentences are logical statements?

(i) Red Fort is in Delhi.

(ii) Where are you going?

(iii) Sarika is not keeping good health.

(iv) What is your opinion about modern movies?

(v) What is your name?

3. Find out which are statements and which are not?

(i) Two non-empty sets have always a non-empty intersection.

(ii) The real number n is less than 2.

(iii) Two individuals are always related.

4. Write down the truth value of the following statements.

(i) A triangle one of whose vertices lies on a circle and whose side opposite to this vertex is a diameter of the circle is a right angled triangle.

(ii) There is always a real root for any quadratic equation.

5. The area of a trapezium is given by the formula $A=\frac{1}{2}(B+b)\times H$, where A is number of sq. units in area, B the number of units in one base, b the number of units in the other base and H the number of units in the height. Find the truth set of b if A = 40, B = 6, and H = 8.

6. Write the truth set of the following open sentences, using quantifier method.

(i) $x+7=16$; (ii) $x+4=11$;

(iii) $2x+7=2\left(x+\frac{7}{2}\right)$; (iv) $5x+2=5(x+2)$.

Answers:

1. (i) Statement, Truth value F; (ii) Statement, Truth value T;

(iii) Statement, Truth value T; (iv) Statement, Truth value T;

(v) Not a statement; (vi) Not a statement;

(vii) Statement, Truth value T; (viii) Statement, Truth value T;

(ix) Not a statement; (x) Statement, Truth value F.

2. (i) The given sentence is depicting a fact and is true, without exceptions.

$\therefore$ It is a statement.

(ii) The given sentence cannot be judged as true or false.

$\therefore$ It is not a statement.

(iii) The given sentence can be judged as true or false, because Sarika would be either in a good health or in a bad health.

$\therefore$ It is a statement.

(iv) The given sentence cannot be judged as true or false.

$\therefore$ It is not a statement.

(v) The given sentence cannot be judged as true or false.

$\therefore$ It is not a statement.

3. (i) The intersection of two non-empty sets may or may not be non-empty. The sets A = {1, 2}, B = {3}, C = {1, 4} are non-empty sets and $A \cap B = \phi$, $A \cap C = \{1\} \neq \phi$.

$\therefore$ The given sentence may or may not be true and would depend upon the sets

The given sentence can be judged as a false assertion. Since we have been able to judge given sentence as true or false.

$\therefore$ It is a sentence.

4. (i) The given statement is true because angle inscribed in a semicircle is always a right angle.

The truth value of the statement is T.

(ii) The given statement is not true, because the roots of the quadratic equation $x^2 + 1 = 0$ are i and $-i$ and neither of which is a real number. The given statement may be true for certain equations like $x^2 - 4 = 0$, but such equations are not enough to declare the given statement as true.

$\therefore$ The statement is false. The truth value of the statement is F.

Negation of a Statement:

Let p be any statement. Then the statement expressing denial of p is called negation of p. negation of p is formed by writing it is false…..before p. Negation is formed by inserting the word 'not' p.

Negation of statement p is denoted by $\sim p$.

For example

(i) Let p be the statement "Ram is intelligent".
Then $\sim p$ is the statement "Ram is not intelligent".

(ii) Let p be the statement that 2 + 3 = 6.
Then $\sim p$ is statement "It is false that 2 + 3 = 6".

Truth value of $\sim p$

Ruth value of $\sim p$ is always opposite to that p. If p is true then $\sim p$ is false. If p is false then $\sim p$ is true. As p is a statement, therefore, it can be either true or false, but cannot be both. The statement $\sim p$ is also sometimes called *not p statement*.

If p is any statement then the truth value of $\sim p$ is assumed up in the following table.

Truth Table for $\sim p$

p	$\sim p$
T	F
F	T

Example 1. *Write the negation of the following statements:*

(i) Chennai is the capital of Tamil Nadu.
(ii) $\sqrt{2}$ is not a complex number.
(iii) All triangles are not equilateral triangles.
(iv) The number 2 is greater than 7.
(v) Every natural number is an integer.

Solution: Negation of the given statements are:

(i) Chennai is not the capital of Tamil Nadu.
(ii) $\sqrt{2}$ is a complex number.
(iii) All triangles are equilateral triangles.
(iv) The number 2 is not greater than 7.
(v) Every natural number is not an integer.

Example 2. *Are the following pair of statements negation of each other?*

(i) The number x is not a rational number.
The number x is not an irrational number.

(ii) The number x is not a rational number.
The number x is an irrational number.

Solution: (i) The negation of the statement:

"The number x is not a rational number" is:

The number x is a rational number.

The second statement is the same: x is not irrational. Therefore, those statements are negation of each other.

(ii) The negation of the statement:

The number x is a rational number is:

The number x is not a rational number or we can say x is an irrational number. The second statement is same. Therefore, they are not negation of each other.

Simple and Compound Statements:

If the truth value of a statement does not explicitly depend on any other statement, then it is called a ***simple statement***. Simple statement cannot be sub-divided into simpler statements.

The statements: (i) 3 is a prime number, (ii) The set of integers is an infinite set, etc. are simple statements.

A ***compound statement*** is a combination of two or more simple statements connected by the words 'and', 'or', etc. It can be sub-divided into two or more simple (or component) statements.

Example 1. *Find the component statement of the following compound statements and check whether they are true or false.*

(i) The number 3 is prime or it is odd.
(ii) All integers are positive or negative.
(iii) 100 is divisible by 3, 11 and 5.

Solution:

(i) p: The number 3 is prime.
q: The number 3 is odd.
p, q are connected by or
It is true.

(ii) p: All integers are positive.
q: All integers are negative.
p, q are connected by the word or p and q both are false.

(iii) p: 100 is divisible by 3.
q: 100 is divisible by 11.
r: 100 is divisible by 5.
p is false, q is false, r is true.

p and q and r is a false statement.

Example 2. *Find the component statement of the following and check whether they are true or not in (i), (iii) and (iv):*

(i) A rhombus is a parallelogram and its four sides are equal.
(ii) A student who has taken Mathematics or Computer Science can go for MCA.
(iii)All rational numbers are positive or negative.
(iv)30 is a multiple of 2, 3, and 5.

Solution: (i) The component statements are

p : A rhombus is a parallelogram.

q : A rhombus has its four sides equal.

p, q are both true.

(ii) The component statements are

p : A student who has taken Mathematics can go for MCA.

q : A student who has taken Computer Science can go for MCA.

(iii) The component statements are

p : All rational numbers are positive.

q : All rational numbers are negative.

p, q are both false.

(iv) The component statements are

p : 30 is a multiple of 2.

q : 30 is a multiple of 3.

r : 30 is a multiple of 5.

p, q, r are all true.

Exercise 14.2

1. Write the negation of the following statements and check whether the resulting statements are true:

(i) Australia is a continent.

(ii) There does not exist a quadrilateral which has all its sides equal.

(iii) Every natural number is greater than 0.

(iv) The sum of 3 and 4 is 9.

2. Write the negation of the following statements:

(i) p: For every real number x, $x^2 > x$.

(ii) q: There exist a rational number x such that $x^2 = 2$.

(iii) r: All birds have wings.

(iv) s: All students study Mathematics at the elementary level.

3. Find the components of the following compound statements and, check whether they are true or false in (i), (ii), (iii), (v), (vi)- (viii), (x)- (xi).

(i) $\sqrt{5}$ is an irrational number and 5 is a rational number.

(ii) (a) $7 > 6$ and $9 < 10$; (b) $\sqrt{3} > 3$ and $\sqrt{16} < 3$.

(iii) It rains and it is cold.

(iv) Shyam secured 80 marks in Physics and 70 marks in Chemistry.

(v) 3 is a natural number and it is a prime number.

(vi) i is a real number and $2+3i$ is a rational number where $i = \sqrt{-1}$.

(vii) A square is a rectangle and its diagonals are equal.

(viii) All rational numbers are real and all real numbers are complex.

(ix) It is hot and I take a cold drink.

(x) 15 is a multiple of 3 and 5.

(xi) 70 is divisible by 3, 4, and 11.

Answers:

1. (i) Australia is not a continent. False.

(ii) There exist a quadrilateral which has all its sides equal. True.

(iii) It is false that every natural number is greater than 0. False.

(iv) The sum of 3 and 4 is not equal to 9. True.

2. (i) $\sim p$: There exist a real number x such that $x^2 > x$.

(ii) $\sim q$: For all real numbers x, $x^2 \neq 2$.

(iii) $\sim r$: There exist a bird which has no wings.

(iv) $\sim s$: There exist a student who does not study Mathematics at the elementary level.

3. (i) p: $\sqrt{5}$ is an irrational number, q: 5 is a rational number; True.

(ii) (a) $p: 7 > 6$, $q: 9 < 10$; True. (b) p: $\sqrt{3} > 3$, $q: \sqrt{16} < 3$; False.

(iii) p: It rains, q: It is cold; True.

(iv) p: Shyam secured 80 marks in Physics, q: Shyam secured 70 marks in Chemistry.

(v) $p: 3$ is a natural number, $q: 3$ is a prime number; true.

(vi) $p: i$ is a real number, $q: 2+3i$ is a rational number; False.

(vii) p: A square is a rectangle, q: Two diagonals of a square are equal; True.

(viii) p: All rational numbers are real, q: All real numbers are complex; True.

(ix) p: It is hot, q: I take a cold drink.

(x) $p: 15$ is a multiple of 3, $q: 15$ is a multiple of 5; True.

(xi) $p: 70$ is divisible by 3, $q: 70$ is divisible by 4, $r: 70$ is divisible by 11; False.

Logical Connectives:

Some connecting words are used to form compound statements. These connecting words are 'and', 'or', 'if-then', 'only if' and 'if and only if'. These are called ***connectives***.

The word 'And':

Any two simple statements can be combined by using the word 'and' to form a compound statement, which may be true or false.

For example, consider the two statements:

p :5 is prime number.

q :5 is a factor of 10, then the compound statement formed by 'and' is

r :5 is prime number and it is a factor of 10.

We see that p and q are both true and compound statement r is also true.

Consider the compound statement connected by 'and':

'77 is divisible by 5, 7, and 11'

The component statements of this statement are:

p :77 is divisible by 5.

q :77 is divisible by 7.

r :77 is divisible by 11.

We note that the first statement is false, whereas the other two statements are true.

The above discussion suggests us the following rules regarding the connective 'and'.

- The compound statement with 'and' is true if all its component statements are true.
- The compound statement with 'and' is false if any or all of its component statements is false.

The word 'or':

Any two simple statements can be combined by using the word 'or' to form a compound statement, which may be true or false. 'or' is used in two different senses: (i) ***'exclusive or'*** and (ii) ***'inclusive or'***.

Exclusive 'Or':

In a compound statement "p *or* q", if exactly one of the two alternatives occur, i.e., component statements p *or* q, but not both occur, then connecting word "or" is called ***exclusive or***.

Inclusive "Or":

In a compound statement "*p or q*", if at least one of the two alternatives occur, i.e., component statements *p or q*, or both occur, then connecting word "or" is called ***inclusive or***.

For example, consider the statement: Two lines intersect at a point or are parallel. Here, "or" is exclusive because it is not possible for two lines to intersect and parallel together.

Consider now the following statement:

The school is closed if it is a holiday or a Sunday. Here "or" is inclusive since school is closed on holiday as well as Sunday.

Rule for the Compound Statement with "or":

- ❖ A compound statement with 'or' is true when one component statement is true or both the component statement are true.
- ❖ A compound statement with 'or' is false when both the component statements are false.

For example,

(i) p or q :Two lines intersect at a point or they are parallel.

The compound statements are

p : Two lines intersect at a point.

q : Two lines are parallel.

Clearly, when p is true q is false and when q is true p is false. Therefore, the compound statement ' p or q ' is true.

(ii) p or q :39 is divisible by 5 or 6.

The compound statements are

p : 39 is divisible by 5.

q : 39 is divisible by 6.

Both p and q are false. Therefore, the compound statement ' p or q ' is false.

Quantifiers:

In Mathematical statements, we often find phrases like "There exists" and "For all" (or "For every"). These two phrases are called ***quantifiers***.

For example, consider the statement:

p :There exists a rectangle whose all sides are equal.

This means that there is at least one rectangle whose all sides are equal.

Consider another statement:

q : For every prime number p , $\sqrt{p}$ is an irrational number.

This means that if S denotes the set of all prime numbers, then for all the members p of the set S, $\sqrt{p}$ is an irrational number.

Consider the following two statements:

1. For every positive number x, there exists a positive number y such that $y < x$.
2. There exists a positive number such that for every positive number x, we have $y < x$.

Although these statements may look similar, they do not say the same thing. In fact, (1) is true whereas (2) is false.

Example 1. *For each of the following compound statements first identify the connecting words and then break it into component statements:*

(i) All rational numbers are real and all numbers are not complex.
(ii) Square of an integer is positive or negative.
(iii) The sand heats up quickly in the Sun and does not cool down fast at night.
(iv) $x = 2$ *and* $x = 3$ *are the roots of the equation* $3x^2 - x - 10 = 0$.

Solution:

(i) Connecting word "AND"
p: All rational numbers are real.
q: All numbers are not complex.

(ii) Connecting word "OR"
p: The square of an integer is positive.
q: The square of an integer is negative.

(iii) Connecting word "AND"
p: The sand heats up quickly in the Sun.
q: The sand does not cool down fast at night.

(iv) Connecting word "AND"
p: $x = 2$ is the roots of the equation $3x^2 - x - 10 = 0$.
q: $x = 3$ is the roots of the equation $3x^2 - x - 10 = 0$.

Example 2. *Identify the quantifier in the following statements and write the negation of the statement:*

(i) There exists a number which is equal to its square.
(ii) For every real number x, x *is less than* $x+1$.
(iii) There exists a capital for every state of India.

Solution:

(i) Quantifier: There exists
p: There exists a number which is equal to its square.
$\sim p$: There does not exists a number which is equal to its square.

(ii) Quantifier: For every
p: For every real number x, x is less than $x+1$.
$\sim p$: For every real number x, x is not less than $x+1$.

(iii) Quantifier: There exists

p : There exists a capital for every state of India.

$\sim p$: There does not exists a capital for every state of India.

Example 3. *State whether the "OR" used in the following statements is exclusive or inclusive. Give reasons for your answer.*

(i) Sun rises or Moon sets.
(ii) To apply for a driving licence, you should have a ration card or passport.
(iii) All integers are positive or negative.

Solution:

(i) When Sun rises the Moon sets. One of the happenings will take place.
$\therefore$ Here "OR" is exclusive.
(ii) To apply for a driving licence either a ration card or passport or both can be used.
$\therefore$ "OR" used here is inclusive.
(iii) All integers are positive or negative.
An integer cannot be both positive and negative at a time.
$\therefore$ Here "OR" is exclusive.

Exercise 14.3

1. For each of the following compound statements first identify the connecting words and then break it into component statements. Also, write the negation of the compound statements.

(i) x is a rational number or x is irrational.

(ii) Sky is blue and $4<6$.

(iii) Ravi is tall and slim.

(iv) Red Fort is in Delhi or in Agra.

2. Identify the quantifier in the following statements and write the negation of the statements.

(i) There exists a quadrilateral which has equal sides.

(ii) For every real number x, x is greater than $x-1$.

(iii) There exists a real root of quadratic equation.

3. State whether "OR" used in the following statements is "exclusive" or "inclusive":

(i) All real numbers are rational or irrational.

(ii) Coin when tossed comes up with head or tail.

(iii) A card drawn from a pack of cards is either black or king.

(iv) A triangle is equilateral or isosceles.

4. Check whether the following pair of statements are negation of each other. Give reasons for your answer.

(i) $x+y=y+x$ is true for every real numbers x and y.

(ii) There exists real numbers x and y for which $x+y=y+x$.

Answers:

1. (i) Connecting word "OR"

p: x is a rational number. q: x is irrational number.

$\sim p$: x is not a rational number,

$\sim q$: x is not an irrational number.

(ii) Connecting word "AND"

p: Sky is blue. q: $4<6$.

$\sim p$: It is false that Sky is blue.

$\sim q$: It is false that $4<6$.

(iii) Connecting word "AND"

p: Ravi is tall. q: Ravi is slim.

$\sim p$: Ravi is not tall.

$\sim q$: Ravi is not slim.

(iv) Connecting word "OR"

p: Red Fort is in Delhi. q: Red Fort is in Agra.

$\sim p$: Red Fort is not in Delhi.

$\sim q$: Red Fort is not in Agra.

2. (i) Quantifier: There exists

There exists a quadrilateral which does not have equal sides.

(ii) Quantifier: For every

For every real number x, it is false that x is greater than $x-1$.

(iii) Quantifier: There exists

There exists no real root of quadratic equation.

3. (i) Exclusive; (ii) Exclusive; (iii) Inclusive; (iv) Inclusive.

4. Statements (i) and (ii) are not the negation of each other.

Implications:

If two statements are combined by phrases (or connectives) "if then", "only if" and "if and only if" , then these phrases (or connectives) are called ***implications***.

If p and q are two statements forming the implication "if p then q", then we denote this implication by "$p \Rightarrow q$".

For example,

If he is strong, then he will he will fight.

If $a = 9$, then $a^2 = 81$.

If it rains, then the atmospheric humidity increases.

are implications

We shall now see how the truth value of an implication "$p \Rightarrow q$" depends upon the truth value of the statements p and q.

(i) Let p be the statement:
p: Number N = 728 is divisible by 4.
and q be the statement
q: Last two digits of N are divisible by 4.
Clearly, p and q both are true.
Now $p \Rightarrow q$: If the number 728 is divisible by 4, then 28 is divisible by 4.
Clearly, $p \Rightarrow q$ is also true.
Thus, if p and q are true, then $p \Rightarrow q$ is also true.

(ii) Let p be the statement:
p: Number N = 728 is divisible by 4.
and q be the statement
q: Last two digits of N are not divisible by 4.
Clearly, p is true and q is false.
Now $p \Rightarrow q$: If the number 728 is divisible by 4, then 28 is not divisible by 4.
Clearly, $p \Rightarrow q$ is false.
Thus, if p is true and q is false, then $p \Rightarrow q$ is false.

(iii) Let p be the statement:
p: Number N = 728 is not divisible by 4.
and q be the statement
q: Last two digits of N are divisible by 4.
Clearly, p is false and q is true.
Now $p \Rightarrow q$: If the number 728 is not divisible by 4, then 28 is divisible by 4.
The statement $p \Rightarrow q$ is taken to be true though common sense cannot help us to infer so. This assumption is made in order to be consistent with the assumption made.

Thus, if p is false and q is true, then $p \Rightarrow q$ is true.

(iv) Let p be the statement:

p: Number N = 728 is not divisible by 4.

and q be the statement

q: Last two digits of N are not divisible by 4.

Clearly, both p and q are false.

Now $p \Rightarrow q$: If the number 728 is not divisible by 4, then 28 is not divisible by 4.

The statement $p \Rightarrow q$ is true.

Thus if both p and q are false, then $p \Rightarrow q$ is true. On the basis of the above discussion, we conclude :

The implication "if p then q" is always true except when p is true and q is false.

Example 1. *Let p denotes "It rains" and q denotes "The atmospheric humidity increases". Write the following statements in symbolic form:*

(i) Atmospheric humidity increases only if it rains.
(ii) Sufficient condition for it to rain is that atmospheric humidity increases.
(iii) Necessary condition for it to rain is that atmospheric humidity increases.
(iv) Whenever atmospheric humidity increases it rains.

Solution:

(i) Statement tells 'q only if p'

$\therefore$ in symbolic form we can write the statement as $q \Rightarrow p$.

(ii) It shows that q is sufficient condition for p.

Therefore, statement in symbolic form is $q \Rightarrow p$.

(iii) It shows that q is sufficient condition for p.

Therefore, statement in symbolic form is $p \Rightarrow q$.

(iv) Whenever atmospheric humidity increases it rains is equivalent to

If atmospheric humidity increases, then it rains.

In symbolic form, statement is $q \Rightarrow p$.

Converse Statement:

If p and q are two statements, then the converse of the implication "if p, then q" is "if q, then p". For example, converse of the statement:

If a number x is odd, then x^2 is also odd.

Converse of the statement is: If x^2 is odd, then x is also odd.

Contrapositive Statement:

If p and q are two statements, then the contrapositive of the implication "if p, then q" is "if not q, then not p". For example, contrapositive of the statement:

If a number is divisible by 4, then it is divisible by 2.

Contrapositive of the statement is: If a number is not divisible by 2, then it is not divisible by 4.

"If and Only If" Implication:

If p and q are two statements, then the compound statement $p \Rightarrow q$ and $q \Rightarrow p$ is called if and only if implication and is denoted by $p \Leftrightarrow q$ having the following equivalent forms:

(i) p if and only if q;
(ii) q if and only if p;
(iii) p is necessary and sufficient condition for q and vice-versa;
(iv) $p \Leftrightarrow q$.

For example, consider the statement:

A rectangle is a square if and only if all its four sides are equal.

This is "if and only if" implication with the component statements

p: If a rectangle is a square, then all its four sides are equal

q: If all the four sides of a rectangle are equal, then the rectangle is a square.

Truth Value of "if and only if"

Statement with "if and only if" is true when

(i) p is true, q is true;
(ii) p is false, q is false.

Example 2. *Rewrite the following statement with "if-then" in five different ways conveying the same meaning:*

If a natural number is odd, then its square is also odd.

Solution: (i) A natural number is odd implies that its square is odd.

(ii) The natural number is odd only if its square is odd.

(iii)If the square of a natural number is not odd, then the natural number is also not odd.

(iv)For a natural number to be odd, it is necessary that its square is odd.

(v)For the square of a natural number to be odd, it is sufficient that the number is odd.

Example 3. *Write the contrapositive and converse of the following statements:*

(i) If x is a prime number, then x is odd.
(ii) If the two lines are parallel, then they do not intersect in the same plane.
(iii) Something is cold implies that it has low temperature.
(iv) You cannot comprehend geometry if you do not know how to reason difficulty.

(v) *x is an even natural number implies that x is divisible by 4.*

Solution: (i) ***Contrapositive Statement:*** If a number x is not odd, then x is not a prime number.

Converse Statement: If x is odd, then x is a prime number.

(ii) ***Contrapositive Statement:*** If two straight lines intersect in a plane, then the lines are not parallel.

Converse Statement: If two lines do not intersect in the same plane, then the two lines are parallel.

(iii) ***Contrapositive Statement:*** If the temperature of something is not low, then it is not cold.

Converse Statement: If something has low temperature, ten it is cold.

(iv) ***Contrapositive Statement:*** If you know how to reason deductively, then you can comprehend geometry.

Converse Statement: If you do not know how to reason deducting, then you cannot comprehend geometry.

(v) ***Contrapositive Statement:*** If x is not divisible by 4, then x is not an even number.

Converse Statement: If x is divisible by 4, then x is an even number.

Example 4. *Write each of the following statements in the form "if-then".*

(i) You get a job implies that your credentials are good.
(ii) The Banana trees will bloom if stays warm for a month.
(iii) A quadrilateral is a parallelogram if its diagonals bisect each other.
(iv) To get an A^+ in the class, it is necessary that you do all exercises of the book.

Solution: (i) If you get a job, then your credentials are good.

(ii) If it stays warm for a month, then Banana trees will bloom.

(iii) If the diagonals of a quadrilateral bisect each other, then it is a parallelogram.

(iv) If you get an A^+ in the class, it is necessary that you do all exercises of the book.

Example 5. *Given statements in (a) and (b), identify the statements as contrapositive or converse of each other:*

(a) If you live in Delhi, then you have winter clothes.
(i) If you do not have winter clothes, then you do not live in Delhi.
(ii) If you have winter clothes, then you live in Delhi.
(b) If a quadrilateral is a parallelogram, then its diagonals bisect each other.
(i) If the diagonals of a quadrilateral do not bisect each other, then the quadrilateral is not a parallelogram.
(ii) If the diagonals of a quadrilateral bisect each other, then the quadrilateral is a parallelogram.

Solution: (a) (i) Contrapositive Statement.

(ii) Converse Statement.

(b) (i) Contrapositive Statement.

(ii) Converse Statement.

Example 6. *State the converse and contrapositive of each of the following statements:*

(i) p *:A positive integer is prime only if it has no divisor other than 1 and itself.*
(ii) q *:I go to a beach whenever it is a sunny day.*
(iii) r *:If it is hot outside, then you feel thirsty.*

Solution: (i) ***Converse:*** If a positive integer has no divisor other than 1 and itself, then it is a prime.

Contrapositive: If a positive integer has no divisor other than 1 and itself, then it is not prime.

(ii) ***Converse:*** If it is a sunny day, then I go to beach.

Contrapositive: If it is not a sunny day, then I do not go to beach.

(iii) ***Converse:*** If you feel thirsty, then it is hot outside.

Contrapositive: If you do not feel thirsty, then it is not hot outside.

Example 7. *Write each of the statements in the form "if* p *then* q *":*

(i) p *:It is necessary to have a password to log on to server.*
(ii) q *:There is a traffic jam whenever it rains.*
(iii) r *:You can access the website if you pay a subscription fee.*

Solution: (i) If you log on to server, then you have a password.

(ii) If it rains, then there is a traffic jam.

(iii) If you pay a subscription fee, then you can access the website.

Example 8. *Rewrite each of the following statements in the form "* p *if and only if* q *":*

(i) p *:If you watch television, then your mind is free and if your mind is free, then you watch a television.*
(ii) q *:For you to get an A Grade, it is necessary and sufficient that you do all the home work regularly.*
(iii) r *:If a quadrilateral is equiangular, then it is a rectangle and if a quadrilateral is a rectangle, then it is equiangular.*

Solution: (i) You watch a television if and only if your mind is free.

(ii) You will get grade A if and only if you do all the home work regularly.

(iii) A quadrilateral is equiangular if and only if it is a rectangle.

Exercise 14.4

1. Write the contrapositive of the following statements:

(i) If a number is divisible by 9, then it is divisible by3.

(ii) If you are born in India, then you are a citizen of India.

(iii) If a triangle is equilateral, it is isosceles.

2. Write the converse of the following statements:

(i) If a number n is even, then n^2 is even.

(ii) If you do all the exercises in the book, you get an A grade in the class.

(iii) If two integers a and b are such that $a > b$, then $a - b$ is always a positive integer.

3. For each of the following compound statements, first identify the corresponding component statements. Then check whether the statements are true or not:

(i) If a triangle ABC is equilateral, then it is isosceles.

(ii) If a and b are integers, then ab is a rational number.

4. State the converse and contrapositive of the following statements:

(i) If the number x is rational, then x^2 is also rational.

(ii) If a triangle is equiangular, then it is equilateral.

(iii) If x is an integer, then $x+1$ is also an integer.

(iv) If it is raining, then you cannot go to market.

5. Write each of the statements in the form "if then":

(i) p: It is necessary to get 60% marks to appear in IITJEE test.

(ii) q: There is a traffic jam whenever there are office hours.

(iii) r: You can board a train when you pay its charge.

(iv) s: There is a bomb blast whenever the militants are there.

6. Write each of the statements in the form "if and only if":

(i) If the team wins when players are good or if the players are good team wins.

(ii) You can catch the train if you reach in time and if you catch a train you are there on time.

(iii) It is necessary and sufficient that a triangle is equilateral if the sides are equal.

(iv) The number x is an even number so is the square of it and if x^2 is even, then x is also even.

7. Write the contrapositive and converse of the following statements:

(i) If a number is composite, then it has prime factors.

(ii) If the sum of the digits of a number is divisible by 3, then it is divisible by 3.

(iii) If the sides of a quadrilateral are equal, then it is a rhombus.

(iv) If it is cold today, then the temperature is low.

8. Write each of the following statements in the form "if-then":

(i) The place is hilly, the temperature is low.

(ii) I can go to market only if my car is not in the repair shop.

(iii) A number is rational, implies that it is real number.

(iv) He wears warm clothes it is sufficient that it is cold.

(v) He will not get admission in an Engineering college if he studies regularly.

9. Identify the statements, contrapositive and converse statements from the following:

(a) "If the water is warm, the he will swim".

(i) If he swims, then water is warm.

(ii) If he does not swims, then water is warm

(b) "If a man is married, then he is happy".

(i) If he is not happy, then he is not married.

(ii) If he is happy, then he is a married man.

10. Given below are two pairs of statements. Combine these two statements using "if and only if":

(i) p : If a rectangle is a square, then all its four sides are equal.

q : If all the four sides of a rectangle are equal, then the rectangle is a square.

(ii) p : If the sum of digits of a number is divisible by 3, then the number is divisible by 3.

q : If a number is divisible by 3, then the sum of its digits is divisible by 3.

Answers:

1. (i) If a number is not divisible by 3, then it is not divisible by 9.

(ii) If you are not a citizen of India, then you were not born in India.

(iii) If a triangle is not isosceles, then it is not equilateral.

2. (i) If a number n^2 is even, then n is even.

(ii) If you get an A grade in the class, then you have done all the exercises of the book.

(iii) If two integers a and b are such that $a-b$ is always a positive integer, then $a > b$.

3. (i) The component statements are given by

p : Triangle ABC is equilateral.

q : Triangle ABC is isosceles.

Since an equilateral triangle is isosceles, we infer that the given compound statement is true.

(ii) The component statements are given by

p : a and b are integers

q : ab is a rational number.

Since the product of two integers is an integer and, therefore, a rational number, the compound statement is true.

4. Converse:

(i) If x^2 is rational, then number x is also rational.

(ii) If a triangle is equilateral, then it is equiangular.

(iii) If $x+1$ is an integer, then x is also an integer.

(iv) If you cannot go to market, then it is raining.

Contrapositive:

(i) If x^2 is not rational, then number x is also not rational.

(ii) If a triangle is not equilateral, then it is not equiangular.

(iii) If $x+1$ is not an integer, then x is also not an integer.

(iv) If you go to market, then it is not raining.

5. (i) p : If you get 60% marks, then you can appear in IITJEE test.

(ii) q : If there is a traffic jam, then there are office hours.

(iii) r : If you pay the charges, then you can board the train.

(iv) s : If there is a bomb blast, then militants are there.

6. (i) The team wins if and only if the players are good.

(ii) You can catch the train if and only if you reach on time.

(iii) A triangle is equilateral if and only if its sides are equal.

(iv) x^2 is an even number if and only if x is an even number.

7. Contrapositive Statement:

(i) If a number does not have prime factors, then it is not a composite number.

(ii) If a number is not divisible by 3, then the sum of its digits is not divisible by 3.

(iii) If a quadrilateral is not a rhombus, then its sides are not equal.

(iv) If the temperature is not low, then it is not cold today.

Converse Statement:

(i) If a number has prime factors, then it is a composite number.

(ii) If number is divisible by 3, then the sum of its digits is also divisible by 3.

(iii) If a quadrilateral is rhombus, then its sides are equal.

(iv) If the temperature is low, then it is cold.

8. (i) If the place is hilly, then the temperature is low.

(ii) If my car is not in repair shop, then I can go to market.

(iii) If a number is rational, then it is real.

(iv) If it is cold, then he wears warm clothes.

(v) If he studies regularly, then he will get admission in Engineering college.

9. (a) (i) Converse; (ii) Contrapositive.

(b) (i) Contrapositive; (ii) Converse.

10. (i) A rectangle is a square if and only if all its four sides are equal.

(ii) A number is divisible by 3 if and only if the sum of its digits is divisible by 3.

Validating Statements:

In this section we shall study the validity of statements, i.e., to check when a particular statement is true and when it is not true. This depends upon which of the special words "and", "or" and which of the implications "if and only if", "if -then" and which of the quantifiers "for every", "there exists", appear in the given statement.

We shall now give some general rules for checking whether a statement is true or not.

Statements with "and": If p and q are Mathematical statements, then in order to show that the statement "p and q" is true, the following steps are followed:

Step I: Show that the statement p is true.

Step II: Show that the statement q is true.

Statements with "or": If p and q are Mathematical statements, then in order to show that the statement "p or q" is true, we must consider the following:

Assuming that p is false, show that q must be true

Or

Assuming that p is false, show that p must be true.

Statements with "if-then": If p and q are two Mathematical statements, then in order to show that the statement "if p then q" is true, we need to show that any one of the following cases is true:

Case I: Assuming that p is true, prove that q must be true. (Direct Method)

Case II: Assuming that q is false, prove that p must be false. (Contrapositive Method)

Statements with "if and only if": In order to prove the statement "p if and only if q", we need to show

1. If p is true, then q is true.
2. If q is true, then p is true.

Validity of Statements by Contradiction:

To check whether a statement p is true or not, sometimes we assume that p is not true, i.e., $\sim p$ is true. Then, we arrive at some result which contradicts our assumption. Therefore, we conclude that p is true. This method is known as *contradiction method*.

Invalidity of Statements by Counter Examples:

In order to show that a statement is false, we may give an example of a situation where the statement is not valid. Such an example is called a counter example. The name itself suggests that this is an example to counter the given statement.

Example 1. Show that the statement:

p: "If x is real number such that $x^3+4x=0$, then $x=0$" is true by

(i) Direct Method; (ii) Method of Contradiction; (iii) Method of Contrapositive.

Solution: (i) Direct Method:

$x^3+4x=0 \qquad \Rightarrow x\left(x^2+4\right)=0$

But $x^3+4x\neq 0, x\in R$, therefore, $x=0$.

(ii) Method of Contradiction:

Let $x\neq 0$ and let it be $x=p, p\in R$, p is a root of $x^3+4x=0$.

$\therefore p^3+4p=0 \qquad \Rightarrow p\left(p^2+4\right)=0$

But $p\neq 0$, also $p^2+4\neq 0$

$\Rightarrow$ So, we get a contradiction. Thus, our assumption is wrong. $\therefore x=0$

(iii) Contrapositive:

Let $x=0$ is not true. Let $x=p\neq 0$.

$\therefore p^3+4p=0$, p being the root of $x^3+4x=0$.

$\Rightarrow p\left(p^2+4\right)=0$. Now, $p\neq 0$. Also, $p^2+4\neq 0$

$\Rightarrow p\left(p^2+4\right)\neq 0$

$\therefore x=0$ is the root of $x^3+4x=0$.

Example 2. *Show that the following statement is true by the method of contrapositive:*

p: *If x is an integer and x^2 us even, then x is also even.*

Solution: Let x be not even, i.e., $x=2n+1$

$\therefore x^2=\left(2n+1\right)^2=4n^2+4n+1=4\left(n^2+n\right)+1$

$4\left(x^2+x\right)+1$ is odd.

i.e., "If q is not true, then p is not true" is proved. Hence the given statement is true.

Example 3. *Given below are two statements:*

p: *25 is a multiple of 5.*

q: *25 is a multiple of 8.*

Write the compound statement, connecting these two statements with "and" and "or". In both cases, check the validity of the compound statement.

Solution: (i) Compound statement with "and":

25 is a multiple of 5 and 8.

This is a false statement since p and q both are not true.

(ii) Compound statement with "or":

25 is a multiple of 5 or it is a multiple of 8.

This is a true statement.

Example 4. *Using the words "necessary and sufficient" rewrite the statement "The integer n is odd if and only if n^2 is odd". Also check whether the statement is true.*

Solution: The necessary and sufficient condition that the integer n be odd is n^2 must be odd. Let p and q denote the statements.

p: The integer n is odd.

q: n^2 is odd.

To check the validity of "p if q", we have to check whether "if p then q" and "if q then p" is true.

Case I: If p, then q

If p, then q is the statement:

If the integer n is odd, then n^2 is odd. We have to check whether this statement is true. Let us assume that n is odd. Then $n = 2k+1$ where k is an integer. Thus $n^2 = (2k+1)^2 = 4k^2 + 4k + 1$.

Therefore, n^2 is one more than an even number and hence is odd.

Case II: If q, then p

If q, then p is the statement

If n is an integer and n^2 is odd, then n is odd.

We have to check whether this statement is true. We check this by contrapositive method. The contrapositive of the given statement is:

If n is an even integer then n^2 is an even integer.

n is even implies that $n = 2k$ for some k. Then $n^2 = 2k^2$. Therefore, n^2 is even.

Example 5. *Check the validity of the statements given below by the method given against it:*

(i) p: The sum of an irrational number and a rational number is irrational (by contradiction method).

(ii) q: If n is a real number with $n > 3$ then $n^2 > 9$ (by contradiction method).

Solution: (i) Let $\sqrt{a}$ be irrational number and b be a rational number.

Their sum $= b + \sqrt{a}$

Let it be not irrational. Therefore, it is a rational number.

$b + \sqrt{a} = \frac{p}{q}$ where p, q are co-prime

$\therefore \quad \sqrt{a} = \frac{p}{q} - b$

L.H.S. $= \sqrt{a} =$ An irrational number

R.H.S. $= \frac{p}{q} - b =$ A rational number

It is a contradiction.

(ii) Let $n > 3$ and $n^2 \leq 9$

Put $n = 3 + a$

$n^2 = 9 + 6a + a^2$

$= 9 + a(6 + a)$

$\therefore \quad n^2 > 9$

Which is a contradiction.

$\Rightarrow$ If $n > 3$, then $n^2 > 9$.

Exercise 14.5

1. Show that the statement

p: If x is a real number such that $x^3 - 1 = 0$, then x is equal to 1 is true by

(i) Direct method; (ii) method of contradiction; (iii) method of contrapositive.

2. Show that the following statement is true by the method of contrapositive.

p: If x is an integer and x^2 odd then x is also odd.

3. Which of the following statements are true and which are false. In each case give a valid reason for saying so.

(i) p: If a transversal intersects two straight lines, then alternate interior angles are equal.

(ii) q: Roots of the equation $x^2 + 1 = 0$ are real.

(iii) r: All integers are either positive or negative.

(iv) s: A chord divides a circle in two segments.

(v) t: $\sqrt{5}$ is a rational number.

4. Verify by the method of contradiction that $\sqrt{7}$ is irrational.

5. Check the validity of the statement given below by contradiction method:

p: The sum of an irrational number and a rational number is irrational.

6. By giving counter example, show that the following statement is not true.

p: If all the angles of a triangle are equal, then the triangle is obtuse angled triangle.

7. Using the words necessary and sufficient rewrite the statement "The integer n is odd if and only if n^2 is odd" and hence check whether it is true or not.

8. Check the validity of the following statements with "and":

(i) 40 is divisible by 4 and 5.

(ii) 3 and 4are the roots of the equation $x^2 - 7x + 12 = 0$.

9. Check the validity of the following statements with "or" and state whether they are true or not:

(i) 48 is a multiple of 8 or 9.

(ii) 65 is a multiple of 5 or 13.

(iii) 80 is a multiple of 6 or 7.

10. Check the validity of the following statements with "if-then" and state whether they are true or not:

(i) If n is a positive number, then $\sqrt{n}$ is irrational.

(ii) If an integer x is divisible by 6, then it is divisible by 3.

(iii) If a natural number x is divisible by 3, then it is divisible by 9.

(iv) If x, y are both integers, then xy is an odd integer.

Answers:

3. (i) False; (ii) False; (iii) False; (iv) True; (v) True.

5. True. **7.** True.

8. (i) True; (ii) True. **9.** (i) True; (ii) True; (iii) False.

10. (i) True; (ii) True; (iii) False; (iv) True.