MATHEMATICS

CLASS XI

(Under AHSEC Curriculum)

Content Developed by-

Dr. Dipak Sarma
&
Dr. Parag Kumar Deb
Associate Professor
Dept. of Mathematics
Cotton College, Assam

Dr. Anjana Bhattacharyya
Associate Professor
Dept. of Mathematics
B. Borooah. College, Assam

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Principle of Mathematical Induction

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CHAPTER
4

PRINCIPLE OF MATHEMATICAL INDUCATION

The two basic processes of reasoning that are commonly used to draw methematical or scientific conclusions are induction and deduction. Induction is the process of reasoning from particular to general and deduction is the process of reasoning from general to particular. Induction begins by observations, and from observation we arrive at some tentative conclusions, called conjectures.

Principle of Mathematical Induction is a fool that can be used to prove a wide variety of conjectures which are true.

Principla of Mathematical Induction

Let P(n) be a statement involving the natural number n. Then P(n) is true for all natural number n if

(i) P(1) is true

(ii) P(m + 1) is true whenever P(m) is true.

Ex 1. If $P(n)$ is the statement " $n(n+1)(n+2)$ is divisible by 6", then what is P(3).

Sol[n]: $P(n): n(n+1)(n+2)$ is divisible by 6

Therefore

$P(3): 60$ is divisible by 6

Ex. 2. If $P(n)$ is the statement "$10n+3$ is prime", then show that $P(1)$ and $P(2)$ are true but $P(3)$ is not true.

Sol[n]: Given

$P(n): 10n+3$ is prime

$P(1): 10.1+3=13$ is prime, which is true.

$P(2): 10.2+3=23$ is prime which is true.

$P(3): 10.3+3=33$ is prime, which is not true.

So, $P(1)$, $P(2)$ are true, but $P(3)$ is not true.

Ex. 3 Let $P(n)$ be the statement

$2^n \geq 3n$. Show that if $P(m)$ is true, then $P(m+1)$ is also true.

Sol[n] :

Given $P(n): 2^n \geq 3n$

$P(m): 2^m \geq 3m$ is true.

Then,

$2^{m+1} = 2^m.2$

$\geq 3m.2$

$= 3(m+1)+3m$

$\Rightarrow 2^{m+1} \geq 3(m+1)$ As $3m \geq 0$

$\Rightarrow P(m+1)$ is true.

Example 4. Use Principle of mathematical induction to prove that

(i) $1^3+2^3+3^3+..................+n^3 = \left(\frac{n(n+1)}{2}\right)^2$

(ii) $1+\frac{1}{1+2}+\frac{1}{1+2+3}+...........+\frac{1}{1+2+3+....+n} = \frac{2n}{n+1}$

Sol^n :

(i) Let the given statement be P(n) i.e

P(n) : $1^3+2^3+3^3+..................+n^3 = \left(\frac{n(n+1)}{2}\right)^2$

For $n=1$

$P(1):1^3 = \left(\frac{1(1+1)}{2}\right)^3$

$\Rightarrow 1=1$ which is ture.

Let P(m) is true for some positive integer m i.e

$1^3+2^3+3^3+..................+m^3 = \left(\frac{m(m+1)}{2}\right)^3$

We shall now prove that P (m+1) is also ture we have

$1^3+2^3+3^2+..................+m^3+(m+1)^3$

$= \left(\frac{m(m+1)}{2}\right)^2 + (m+1)^3$

$= (m+1)^2\left[\frac{m}{4}+(m+1)\right]$

$= (m+1)^2\frac{(m+2)^2}{4}$

$= \left(\frac{(m+1)(m+2)}{2}\right)^2$

$\Rightarrow$ $P(m+1)$ is ture whenever $P(m)$ is ture. Hence by principle of mathematical induction P(n) is ture for all

$n \in N \cdot$

11. Let the given statement be P(n) i.e.

$$P(n): 1+\frac{1}{1+2}+\frac{1}{1+2+3}+.....+\frac{1}{1+2+3+....+n}=\frac{2n}{n+1}$$

There

$$P(1): 1=\frac{2.1}{1+1}$$

$\Rightarrow 1=1$ which is true.

Let P(m) be true for any positive integer m

i.e. $1+\frac{1}{1+2}+\frac{1}{1+2+3}+...+\frac{1}{1+2+3+...+m}=\frac{2m}{m+1}$

We shall prove that $P(m+1)$ is also true.

We have

$$1+\frac{1}{1+2}+\frac{1}{1+2+3}+....+\frac{1}{1+2+3+....+m}+\frac{1}{1+2+3+....+(m+1)}$$

$$=\frac{2m}{m+1}+\frac{1}{\frac{(m+1)(m+2)}{2}} \qquad \because 1+2+3+....+(m+1)=\frac{(m+1)(m+1+1)}{2}$$

$$=\frac{2}{m+1}\left[m+\frac{1}{m+2}\right]$$

$$=\frac{2}{m+1}\times\frac{m^2+2m+1}{m+2}$$

$$=\frac{2(m+1)^2}{(m+1)(m+2)}$$

$$=\frac{2(m+1)}{m+2}$$

$\Rightarrow P(m+1)$ is true if $P(m)$ is true.

Hence by principle of mathematical induction P(n) is ture for all $n \in N$

(iii) Let the given statement be P (n) i.e

$$P(n): \left(1+\frac{3}{1}\right)\left(1+\frac{5}{4}\right)\left(1+\frac{7}{9}\right)........\left(1+\frac{2n+1}{n^2}\right)=(m+1)^2$$

For $n=1$

$$P(1): \left(1+\frac{3}{1}\right)=(1+1)^2$$

$\Rightarrow 4 = 4$ which is true.

Let $P(m)$ is true for some positive integer m i.e.

$$P(m): \left(1+\frac{3}{1}\right)\left(1+\frac{5}{4}\right)\left(1+\frac{7}{9}\right)............\left(1+\frac{2m+1}{m^2}\right) = (m+1)^2$$

We then prove that $P(m+1)$ is also true.

We have

$$\left(1+\frac{3}{1}\right)\left(1+\frac{5}{4}\right)\left(1+\frac{7}{9}\right).......\left(1+\frac{2m+1}{m^2}\right)\left(1+\frac{2m+3}{(m+1)^2}\right)$$

$$= (m+1)^2\left(1+\frac{2m+3}{(m+1)^2}\right)$$

$$= (m+1)^2 + 2m + 3$$

$$= m^2 + 4m + 4$$

$$= (m+2)^2$$

$\therefore$ $P(m+1)$ is true whenever P(m) is true. Hence by principle of mathematical induction, P(n) is true for all $n \in N$,

Ex. 5. Using principle of mathematical induction prove that n(n+1) (n+5) is a multiple of 3.

Sol[n] : Let $P(n)$ be the given statement i.e.

$P(n): n(n+1)(n+5)$ is a multiple of 3

For $n = 1$,

$P(1): 1(1+1)(1+5) = 2.6 = 12$ is a multiple of 3

Which is true,

Next let $P(m)$ is true for any positive integer m. i.e. $m(m+1)(m+5)$ is a multiple of 3.

So, Let $m(m+1)(m+5) = 3k$ .... (i)

We shall then prove that $P(m+1)$ is true .

Here

$$(m+1)(m+2)(m+6) = (m+1)\{(m+1)+1\}\{(m+5)+1\}$$

$$= m\{(m+1)+1\}(m+5+1)+1(m+1+1)(m+5+1)$$

$$= m\{(m+1)(m+5)+(m+1)+(m+5)+1\}+(m+2)(m+6)$$

$$= m(m+1)(m+5)+m^2+m+m^2+5m+m+m^2+8m+12$$

$$= 3k + 3m^2 + 15m + 12 \text{ using(i)}$$

$= 3\left(k + m^2 + 5m + 4\right)$

Which is a multiple of 3.

Hence $P(m+1)$ is true if $P(m)$ is true

Therefore by principle of mathematical induction $P(n)$ is true for any positive integer n.

Ex. 6 By induction prove that $(1+x)^n \geq 1 + nx$ for all $n \in N$ and $x > -1$

Sol^n : Let

$P(x) : (1+x)^n \geq 1 + nx, n \in N, x > -1$

For $n = 1$,

$P(1) : (1+x)^1 \geq 1 + x$

$\Rightarrow 1 + x \geq 1 + x$, which is true

Let $P(m)$ is true for some positive intiger is i.e. $(1+x)^m \geq 1 + mx$ .... (i)

Given . $x > -1 \Rightarrow (x+1) > 0$

Multiplying both sides of (i) by (1 + x)

$(1+x)^m (1+x) \geq (1+mx)(1+x)$

$\geq 1 + (m+1)x$

$\left[\therefore m \in N \right.$ & $x^2 \geq 0$ for every $x \in R$, so $mx^2 \geq 0$

$\Rightarrow P(m+1)$ is true it $P(m)$ is true.

Hence by induction P(n) is true for all $n \in N$

Ex. 7. Prove using principle of mathematical induction that $x^{2x} - y^{2n}$ is divisible by x + y.

Sol^n : Let the statement be p(n) i.e.

$P(n) : x^{2n} - y^{2n}$ is divisible by x + y

For n = 1

$P(1) : x^2 - y^2$ is divisible by $x + y$ which is true.

Let $P(m)$ is true, where m is any positive integer.

i.e $x^{2m} - y^{2m}$ is divisible by x + y

or $x^{2m} - y^{2m} = (x+y)\, f(x, y)$ .... (i)

Where $f(x, y)$ is a polynomial in x and y

(i) $\Rightarrow x^{2m} = (x+y) f(x, y) + y^{2m}$ .... (ii)

We shall now prove that P(m+1) is true .

Here $x^{2(m+1)} - y^{2(m+1)} = x^{2m}.x^2 - y^{2m}.y^2$

$= \{(x+y) f(x, y) + y^{2m}\} x^2 - y^{2m} y^2$ .... using (2)

$= (x+y) f(x, y) x^2 + y^{2m}(x^2 - y^2)$

$= (x+y)[x^2 f(x+y) + (x-y) y^{2m}]$

Which is divisible by x + y

$\Rightarrow P(m+1)$ is ture if $P(m)$ is true.

Therefore by principle of mathematical induction P(n) is true for all $n \in N$

Ex. 8. Prove by induction that $n^2 + n$ is an even natural number.

Sol^n : Let the given statement be $P(n)$ is $P(n)$: $n^2 + n$ is an even natural number

Then $P(1)$: $1^2 + 1 = 2$ is an even natural number.

which is true.

Let P(m) be true for some positive integer m.

i.e. P(m) : $m^2 + m$ is an even natural number

$\therefore$ $m^2 + m = 2k$ .... (i) where $k \in N$

We shall now show that $P(m+1)$ is true

Here

$(m+1)^2 + (m+1) = m^2 + 2m + 1 + m + 1$

$= m^2 + m + 2(m+1)$

$= 2k + 2(m+1)$ using (i)

$= 2(k + m + 1)$ which is an even natural number

$\therefore$ P (m + 1) is true

So, by principle of induction, P (n) is true for all $n \in N$.

Example 9. If p be a natural number then prove that $p^{n+1} + (p+1)^{2n-1}$ is divisible by $p^2 + p + 1$

Sol^n: Let P(n) be the statement

$p^{n+1} + (p+1)^{2n-1}$ is divisible by $p^2 + p + 1$

Then for $n=1$,

$P(1)=p^{1+1}+(p+1)^{2.1-1}=p^2+p+1$ is divisible by p^2+p+1 which is ture.

$\therefore$ $P(1)$ is true.

Let, $P(m)$ be true

i.e $p^{m+1}+(p+1)^{2m-1}$ is divisible by p^2+p+1

$\Rightarrow$ $p^{m+1}+(p+1)^{2m-1}=(p^2+p+1)f(p)$ .... (i)

Where $f(p)$ is a polynomial in p.

For $P(m+1): p^{m+1+1}+(p+1)^{2(m+1)-1}$

$=p^{m+2}+(p+1)^{2m-1}.(p+1)^2$

$=p^{m+2}+\left[(p^2+p+1)f(p)-p^{m+1}\right](p+1)^2$ using (i)

$=p^{m+1}p-p^{m+1}(p+1)^2+(p+1)^2(p^2+p+1)f(p)$

$=(p^2+p+1)\left[(p+1)^2 f(p)-p^{m+1}\right]$

Which is divisible by p^2+p+1

$\Rightarrow p(m+1)$ is true.

Hence by principle of mathematical induction P(n) is true for all $n \in N$.

Ex. 10. Using principle of mathematical inducation prove that, 3^{2n} when divided by 8, leaves the remainder 1.

Sol^n : Let the statement be

P (n) : 3^{2n} when divided by 8, leaves the remainder 1.

For n = 1

$P(1): 3^2=9$, when divided by 8d, leaves the remainder 1, which is true

$\therefore$ P(1) is true

Let $P(m)$ be true.

i.e $P(m)$: 3^{2m} when divided by 8, leaves the remainder 1.

$\therefore$ $3^{2m}=8k+1$ .... (i) for any integer k.

Then

$P(m+1): 3^{2(m+1)}$

$= 3^{2m}.3^2$

$= 3^{2m}.(8+1)$

$= 8.3^{2m} + 3^{2m}$

$= 8(8k+1)+(8k+1)$ using (1)

$= 8^2 k + 8k + 9$

$= 8(8k + k + 1) + 1$

$= 8(9k + k) + 1$

$= 8k' + 1$

where $k' = 9k + 1$ is again an integer

$\Rightarrow 3^{2(m+1)}$ when divided by 8, leaves the remainder 1.

$\Rightarrow P(m+1)$ is true when $P(m)$ is true

$\therefore$ By induction $P(n)$ is true for all $n \in N$.

Ex.11. Prove, by using principle of mathematical induction that

$$\frac{1}{3.5}+\frac{1}{5.7}+\frac{1}{7.9}+........+\frac{1}{(2n+1)(2n+3)}$$

$$=\frac{n}{3(2n+3)}$$

Sol^n : The given statement is $P(n):\frac{1}{3.5}+\frac{1}{5.7}+\frac{1}{7.9}+........+\frac{1}{(2n+1)(2n+3)}$

$$=\frac{n}{3(2n+3)}$$

For n = 1

$P(1):\frac{1}{3.5}=\frac{1}{3.(2+3)}=\frac{1}{3.5}$ which is true.

$\therefore$ $P(1)$ is true.

Let $P(m)$ is true for any positive integer m.

i.e. $P(m)=\frac{1}{3.5}+\frac{1}{5.7}+\frac{1}{7.9}+........+\frac{1}{(2m+1)(2m+3)}$

$= \frac{m}{3(2m+3)}$ .... (i)

Then

$P(m+1): \frac{1}{3.5} + \frac{1}{5.7} + ... + \frac{1}{(2m+1)(2m+3)} + \frac{1}{(2m+3)(2m+5)}$

$= \frac{m}{3(2m+3)} + \frac{1}{(2m+3)(2m+5)}$

$= \frac{1}{2m+3}\left[\frac{m}{3} + \frac{1}{2m+5}\right]$

$= \frac{1}{2m+3} \times \frac{2m^2+5m+3}{3(2m+5)}$

$= \frac{(2m+3)(2m+1)}{(2m+3)3(2m+5)}$

$= \frac{m+1}{3(2(m+1)+3)}$

$\Rightarrow P(m+1)$ is true.

Hence by principle of mathematical induction $P(n)$ is true for all $n \in N$

Ex. 12. Use mathematical induction to prove that

$1\underline{|1} + 2\underline{|2} + 3\underline{|3} + ... + n\underline{|n} = \underline{|n+1} - 1$

for all $n \in N$

Sol^n : $1\underline{|1} + 2\underline{|2} + 3\underline{|3} + ... + n\underline{|n} = \underline{|n+1} - 1$

When $n = 1$

$P(1): 1\underline{|1} = \underline{|2} - 1$

$\Rightarrow 1 = 1$ which is true

Let $P(m)$ be true.

i.e $1\underline{|1} + 2\underline{|2} + 3\underline{|3} + ... + m\underline{|m} = \underline{|m+1} - 1$

Then

$P(m+1): 1\underline{|1} + 2\underline{|2} + 3\underline{|3} + ... + m\underline{|m} + (m+1)\underline{|m+1}$

$= \underline{|m+1} - 1 + (m+1)\underline{|m+1}$ using (1)

$= \underline{|m+1}(m+1+1)-1$

$= \underline{|m+2}-1$

$\Rightarrow P(m+1)$ is true whenever $P(m)$ is true.

Hence by principle of induction, $P(n)$ is true.

EXERCISE : 4

1. Let $P(n)$ be the statement “ $3^{2n}-1$ is divisible by 8”. what is $P(n+1)$?

2. It $P(n)$ is the statement "$n(n+1)(n+2)$ is an integral multiple of 12”, prove that $P(3)$ and $P(4)$ are true, but $P(5)$ is not true.

3. Using principle of mathematical induction, prove the following for all $n \in N$

(i) $\frac{1}{2}+\frac{1}{4}+\frac{1}{8}+.............+\frac{1}{2^n}=1-\frac{1}{2^n}$

(ii) 1.2.3 + 2.3.4 + 3.4.5 + ...+ n(n+1) (n+2)$=\frac{n(n+1)(n+2)(n+3)}{4}$

(iii) $3x+6x+9x+........+$ to n terms $=\frac{3}{2}n(n+1)x$

(iv) $\left(1-\frac{1}{2}\right)\left(1-\frac{1}{3}\right)\left(1-\frac{1}{4}\right)................\left(1-\frac{1}{n+1}\right)=\frac{1}{n+1}$

(v) $10^{2n-1}+1$ is divisible by 11

(vi) $n^3+(n+1)^3+(n+2)^3$ is a multiple of 9.

(vii) $3^n > n$ for all $n \in N$.

(viii) $(ab)^n = a^n b^n$,

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