MATHEMATICS

CLASS XI

(Under AHSEC Curriculum)

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Name of the textbook

MATHEMATICS

Name of the Chapter

Complex Numbers & Quadratic equations (Chapter 5)

CONTENTS

COMPLEX NUMBERS

A number of the form a + ib, where a and b are real numbers and i is defined as $i^2 = -1$ i.e. $i = \sqrt{-1}$ is called a complex number.

The number $\sqrt{-1}$ denoted by i, is called the imaginary unit, which is taken as the fundamental unit of the set of complex numbers.

A complex number a + ib is generally denoted by the symbol Z. 'a' is called the real part and 'b' is called the imaginary part of Z.

Two complex numbers $Z_1 = a + ib$ and $Z_2 = c + id$ are equal if a = c and b = d.

The number a – ib is called the conjugate of the number Z = a + ib is denoted by $\overline{Z}$.

For example : $Z = 1 + 2i \Rightarrow \overline{Z} = 1 - 2i$

$Z = 1 \Rightarrow \overline{Z} = 1$

$Z = -4i \Rightarrow \overline{Z} = 4i$ etc.

The sum, difference, product and quotient of two complex numbers $Z_1 = a + ib$ and $Z_2 = c + id$ is defined as :

$Z_1 + Z_2 = (a + c) + i\,(b + d)$

$Z_1 - Z_2 = (a - c) + i\,(b - d)$

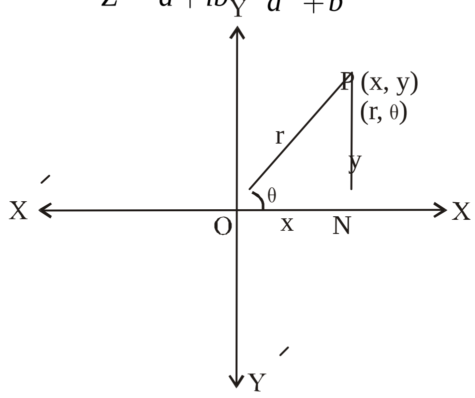
$Z_1 Z_2 = (ac - bd) + i\,(ad + bc)$

$$\frac{Z_1}{Z_2} = \frac{a+ib}{c+id} = \frac{(a+ib)(c-id)}{(c+id)(c-id)} = \frac{(ac-bd)+i\,(bc-ad)}{c^2+d^2}$$

$$= \frac{ac-bd}{c^2+d^2} + i\frac{bc-ad}{c^2+d^2} \quad , \; Z_2 \neq 0$$

Clearly $Z_1 + Z_2, Z_1 - Z_2, Z_1 Z_2, \dfrac{Z_1}{Z_2} \in C,$ where C is the set of complex numbers. For the complex number Z = a + ib, the number –Z = –a – ib is called the additive inverse and the number $\dfrac{1}{Z} = \dfrac{1}{a+ib} = \dfrac{a-ib}{a^2+b^2}$ is called the multiplicative inverse of Z.

Geometrical representation of a complex number.



A complex number Z = x + iy is represented by the point P(x, y) on XY-plane which is called the Z-plane or complex plane or argand plane.

In Fig (1), the complex number Z = x + iy is represented by the point P(x, y).

Join OP.

Let OP = r, ∠PON = θ.

Then from ΔOPN, x = r Cosθ, y = r Sinθ

$\therefore$ Z = x + iy = r (Cosθ + i Sinθ)

This is called polar form of Z and (r, θ) is called the polar co-ordinates of the point P.

Modulus and argument :

Let the complex number Z = x + iy be represented by the point P(x, y) in the argand plane. [Fig. (i)] Then the distance OP is called modulus of Z, denoted by |Z|.

The angle θ made by OP with positive X-axis is called argument or amplitude of Z, denoted by argZ or ampZ.

Here Z = x + iy = r (cosθ + sinθ)

$\therefore$ x = rcosθ , y = rsinθ

which gives $r = \sqrt{x^2 + y^2}$, $\tan\theta = \frac{y}{x}$

Hence $\arg Z = \theta = \tan^{-1}\frac{y}{x}$

argZ has infinite number of values, differing by multiples of 2π. The value of θ satisfying the condition $-\pi < \theta \le \pi$ is known as the principal value of the argument.

Suppose Z = x + iy = a + ib i.e. x = a, y = b

If a > 0, b > 9 then

$$\tan\theta = \frac{y}{x} = \frac{b}{a} = \tan\alpha$$

So that θ = α is the principal value of arZ.

Similarly,

if a > 0, b < 0, then θ = – α is the principal value of arg Z

if a < 0, b > 0, then θ = π – α is the principal value of arg Z

if a < 0, b < 0, then θ = – (π – α) is the principal value of arg Z

Argument of 0 = 0 + i.0 is undefined

For example if Z = 1 + i, $|Z| = \sqrt{2}$, $\arg Z = \pi/4$

If Z = –1 – i, $|Z| = \sqrt{2}$, $\arg Z = -\left(\pi - \pi/4\right) = -3\pi/4$

Properties : For any two complex numbers Z_1 and Z_2

(i) $\overline{Z_1 \pm Z_1} = \overline{Z_1} \pm \overline{Z_1}$

(ii) $\overline{Z_1.Z_2} = \overline{Z_1}.\overline{Z_2}$

(iii) $\overline{\left(\frac{Z_1}{Z_2}\right)} = \frac{\overline{Z_1}}{\overline{Z_2}}$, $Z_2 \ne 0$

(iv) $|Z_1\, Z_2| = |Z_1||Z_2|$

(v) $\left|\frac{Z_1}{Z_2}\right| = \frac{|Z_1|}{|Z_2|}$, $|Z_2| \ne 0$

(vi) $\arg|Z_1\, Z_2| = \arg Z_1 + \arg Z_2$

(vii) $\arg\left(\frac{Z_1}{Z_2}\right) = \arg Z_1 - \arg Z_2$

For a complex number Z

(i) $Z = \overline{Z} \Rightarrow Z$ is real

(ii) $Z + \overline{Z} = 0 \Rightarrow Z$ is purely imeginary

(iii) $Z\,\overline{Z} = |Z|^2$

Quadratic equation :

The solution of the equation $ax^2 + bx + c = 0$, where $a \neq 0$, b, c are real numbers and $b^2 - 4ac < 0$, are available in the system of complex numbers.

The roots are given by

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a} = \frac{-b \pm i\sqrt{4ac - b^2}}{2a}$$

Example-1 : Express the following in the form a + ib

(i) $i^9 + i^{19}$ **(ii)** $(1 - i)^4$ **(iii)** $\left[\left(\frac{1}{3} + i\frac{7}{3}\right) + \left(4 + i\frac{1}{3}\right)\right] - \left(-\frac{4}{3} + i\right)$

Solution :

(i) $i^9 + i^{19} = i^8.i + i^{18}.i$

$= \left(i^2\right)^4.i + \left(i^2\right)^9.i$

$= i + (-1).\,i = 0 = 0 + i.0$

(ii) $(1 - i)^4 = \left\{(1-i)^2\right\} = \left\{1 - 2i + i^2\right\}^2 = (-2i)^2$

$= 4i^2 = -4 = -4 + i.0$

(iii) $\left[\left(\frac{1}{3} + i\frac{7}{3}\right) + \left(4 + i\frac{1}{3}\right)\right] - \left(-\frac{4}{3} + i\right)$

$= \left(\frac{1}{3} + i\frac{7}{3}\right) + \left(4 + i\frac{1}{3}\right) - \left(-\frac{4}{3} + i\right)$

$= \frac{13}{3} + i\frac{8}{3} + \frac{4}{3} - i = \frac{17}{3} + i\frac{5}{3}$

Example-2 : Find the multiplicative inverse of

(i) $\sqrt{5} + 3i$ **(ii)** $-i$

Solution :

(i) Multiplicative inverse of $\sqrt{5}+3i$ is

$$\frac{1}{\sqrt{5} + 3i} = \frac{\sqrt{5} - 3i}{\left(\sqrt{5} + 3i\right)\sqrt{5} - 3i} = \frac{\sqrt{5} - 3i}{5 + 9i^2} = \frac{\sqrt{5} - 3i}{14} = \frac{\sqrt{5}}{14} - \frac{3}{14}i$$

(ii) Multiplicative inverse of $-i$ is

$$\frac{1}{-i} = \frac{-i^2}{-i} = i$$

Example-3 : Evaluate $\left[i^{18} + \left(\frac{1}{i}\right)^{25}\right]^3$

Solution : $\left[i^{18}+\left(\frac{1}{i}\right)^{25}\right]^3$

$=\left[\left(i^2\right)^9+(-i)^{25}\right]^3$

$=\left[-1-i^{25}\right]^3=\left[-1-\left(i^2\right)^{12}.i\right]^3$

$=(-1-i)^3=-\left(1+i^3+3i^2+3i\right)$

$=-(1-i-3+3i)^3=-(-2+2i)=2-2i$

Example-4 : If $x-iy=\sqrt{\frac{a-ib}{c-id}}$, prove that $\left(x^2+y^2\right)=\frac{a^2+b^2}{c^2+d^2}$

Solution : Given

$$x-iy=\sqrt{\frac{a-ib}{c-id}}$$

Replacing –i by i

$$x+iy=\sqrt{\frac{a+ib}{c+id}}$$

$$\therefore \quad (x+iy)(x-iy)=\sqrt{\frac{a+ib}{c+id}}\times\sqrt{\frac{a-ib}{c-+id}}$$

$$\Rightarrow \quad x^2+y^2=\sqrt{\frac{(a+ib)(a-ib)}{(c+id)(c-id)}}$$

$$\Rightarrow \quad \left(x^2+y^2\right)^2=\frac{a^2+b^2}{c^2+d^2}$$

Example-5 : Convert $\frac{1+3i}{1-2i}$ in to polar form

Solution : Let $\frac{1+3i}{1-2i}=\frac{(1+3i)(1+2i)}{(1-2i)(1+2i)}$

$$=\frac{(1-6)+i(3+2)}{1+4}=\frac{-5+5i}{5}=-1+i$$

$$\therefore \quad |Z|=\sqrt{1^2+1^2}=\sqrt{2}$$

$$\tan\alpha=\left|\frac{1}{-1}\right|=1 \Rightarrow \alpha=\frac{\pi}{4}$$

$\therefore \quad \arg Z = \pi - \alpha = \pi - \frac{\pi}{4} = \frac{3\pi}{4}$

$\therefore$ Polar form of $\frac{1+3i}{1-2i}$ is $\sqrt{2}\left(\cos\frac{3\pi}{4} + i\sin\frac{3\pi}{4}\right)$

Example-6 : If α and β are different complex numbers with | β | = 1, then find $\left|\frac{\beta-\alpha}{1-\bar{\alpha}\beta}\right|$

Solution : Let x = $\left|\frac{\beta-\alpha}{1-\bar{\alpha}\beta}\right|$

$\Rightarrow \quad x^2 = \left|\frac{\beta-\alpha}{1-\bar{\alpha}\beta}\right|^2 = \left(\frac{\beta-\alpha}{1-\bar{\alpha}\beta}\right)\overline{\left(\frac{\beta-\alpha}{1-\bar{\alpha}\beta}\right)} \qquad \left[\because Z\bar{Z} = |Z|^2\right]$

$= \frac{(\beta-\alpha)(\bar{\beta}-\bar{\alpha})}{(1-\bar{\alpha}\beta)(1-\alpha\bar{\beta})}$

$= \frac{\beta\bar{\beta} - \beta\bar{\alpha} - \alpha\bar{\beta} + \alpha\bar{\alpha}}{1-\alpha\bar{\beta} - \bar{\alpha}\beta + \alpha\bar{\alpha}\beta\bar{\beta}}$

$= \frac{|\beta|^2 - \alpha\bar{\beta} - \bar{\alpha}\beta + \alpha\bar{\alpha}}{1-\alpha\bar{\beta} - \bar{\alpha}\beta + \alpha\bar{\alpha}|\beta|^2}$

$= 1 \qquad [\because |\beta| = 1 \text{ given}]$

$\Rightarrow \quad x = 1$

$\Rightarrow \quad \frac{\beta-\alpha}{1-\bar{\alpha}\beta} = 1$

Example-7 : Find the number of non zero integral solutions of the equation $|1-i|^x = 2^x$

Solution : $|1-i|^x = 2^x$

$\Rightarrow \quad \left(\sqrt{1^2+1^2}\right)^x = 2^x \Rightarrow 2^{x/2} = 2^x$

$\Rightarrow \quad x/2 = x \Rightarrow x = 0$

$\therefore$ There is no non zero integral solution.

Example-8 : If $\left(\frac{1+i}{1-i}\right)^m = 1$**, then find the least positive integral value of m.**

Solution : $\left(\frac{1+i}{1-i}\right)^m = 1$

$\Rightarrow \quad \left\{\frac{(1+i)(1+i)}{(1-i)(1+i)}\right\}^m = 1$

$$\Rightarrow \quad \left\{\frac{1+2i+i^2}{1+1}\right\}^m = 1$$

$$\Rightarrow \quad \left\{\frac{2i}{2}\right\}^m = 1 \quad \Rightarrow \quad i^m = 1$$

$\therefore$ Least positive integral value of m is 4.

Example-9 : If $|Z_1 + Z_2|^2 = |Z_1|^2 + |Z_2|^2$**, show that** $\dfrac{Z_1}{Z_2}$ is purely imeginary**.**

Solution : Given,

$$|Z_1 + Z_2|^2 = |Z_1|^2 + |Z_2|^2$$

$$\Rightarrow \quad |Z_1 + Z_2|\,|\overline{Z_1 + Z_2}| = Z_1\overline{Z_1} + Z_2\overline{Z_2} \qquad \left[\because\ Z\overline{Z} + |Z|^2\right]$$

$$\Rightarrow \quad (Z_1 + Z_2)\left(\overline{Z_1} + \overline{Z_2}\right) = Z_1\overline{Z_1} + Z_2\overline{Z_2}$$

$$\Rightarrow \quad Z_1\overline{Z_1} + Z_1\overline{Z_2} + Z_2\overline{Z_1} + Z_2\overline{Z_2} = Z_1\overline{Z_1} + Z_2\overline{Z_2}$$

$$\Rightarrow \quad Z_1\overline{Z_2} = -Z_2\overline{Z_1}$$

$$\Rightarrow \quad \frac{Z_1}{Z_2} = -\frac{\overline{Z_1}}{\overline{Z_2}} = -\overline{\left(\frac{Z_1}{Z_2}\right)}$$

$\Rightarrow \quad \left(\dfrac{Z_1}{Z_2}\right)$ is purely imeginary**.**

Example-10 : If $(x+iy)^3 = u + iv$**, then show that** $\dfrac{u}{x} + \dfrac{v}{y} = 4\left(x^2 - y^2\right)$

Solution : Given, $(x+iy)^3 = u + iv$

$$\Rightarrow \quad x^3 + 3x^2(iy) + 3x(iy)^2 + (iy)^3 = u + iv$$

$$\Rightarrow \quad x^3 - 3xy^2 + i\left(3x^2y - y^3\right) = u + iv$$

Equating real and imaginary parts

$$x^3 - 3xy^2 = u \qquad \text{and} \qquad 3x^2y - y^3 = v$$

$$\Rightarrow \quad \frac{u}{x} = x^2 - 3y^2 \qquad \Rightarrow \quad \frac{v}{y} = 3x^2 - y^2$$

$$\therefore \quad \frac{u}{x} + \frac{v}{y} = 4x^2 - 4y^2$$

$= 4\left(x^2 - y^2\right)$ ***Proved.***

Example-11 : Solve the equation

$$x^2+\frac{x}{\sqrt{2}}+1=0$$

Solution : $x^2+\frac{x}{\sqrt{2}}+1=0$

$\Rightarrow \quad \sqrt{2}x^2+x+\sqrt{2}=0$

The solutions of the above equation are given by

$$x = \frac{-1\pm\sqrt{1-4.2}}{2.\sqrt{2}}$$

$$= \frac{-1\pm\sqrt{-7}}{2.\sqrt{2}}$$

$$= \frac{-1\pm i\sqrt{7}}{2.\sqrt{2}}$$

Example-12 : If $p, q\in R$ **and 2 + 3i is a root of the equation** $x^2+px+q=0$**, then find the values of p and q.**

Solution : Roots of the equation $x^2+px+q=0$ are given by

$$x=\frac{-p\pm\sqrt{p^2-4q}}{2}=\frac{-p\pm i\sqrt{4q-p^2}}{2}$$

Given 2 + 3i is a root of the given equation

$\therefore \quad \frac{-p+i\sqrt{4q-p^2}}{2}=2+3i$

$\Rightarrow \quad p+i\sqrt{4q-p^2}=4+6i$

$\Rightarrow \quad p=-4$ and $4q-p^2=36$

$\Rightarrow \quad 4q=36+16=52$

$\Rightarrow \quad q=\frac{52}{4}=13$

$\therefore$ p = –4, q = 13

EXECISE : 5

1. Express the following in the standard form a + ib

(i) $\left(-\sqrt{-1}\right)^{31}$ (ii) $\left(2i-i^2\right)^2+\left(1-3i\right)^3$

(iii) $\frac{5+\sqrt{2}i}{1-\sqrt{2}i}$ (iv) $\frac{1}{\left(2+3i\right)\left(1-2i\right)}$

(v) $\left(-1+\sqrt{3}i\right)^2$ (vi) $(3+2i)^3$

2. Find the multiplicative inverse of

(i) $\sqrt{5}+3i$ (ii) $\cos\theta + i\sin\theta$

3. Find the conjugate of $\frac{1+i}{1-i}-\frac{1-i}{1+i}$

4. Find the real values of x and y if $(x-iy)(3+5i)$ is the conjugate of $-4-16\,\text{i}$

5. Solve the equation $2z=|z|+2i$

6. Prove that the complex number $z=x+iy$ which satisfy the equation $\left|\frac{z-5i}{z+5i}\right|=1$ lies on the x - axis.

7. What region does the inequality $|z-4|<|z-2|$ represent?

8. If $Z_1=2-i$ and $Z_2=-2+i$, find

(i) $\text{Re}\left(\frac{1}{Z_1\bar{Z}_1}\right)$ (ii) $\text{Im}\left(\frac{Z_1Z_2}{\bar{Z}_1}\right)$

9. Find the value of the sum

$\sum_{n=1}^{13}\left(i^n+i^{n+1}\right)$, where $i=\sqrt{-1}$

10. Find the modulus and argument of

(i) $\frac{1}{1+i}$ (ii) $2\left(-1+i\sqrt{3}\right)$

(iii) $i(1+i)$ (iv) -3

11. If z is a nonzero complex number, then what is the amplitude of $Z\bar{Z}$.

12. If $|Z|=2$ and amp $(z)=5\pi/6$, then write Z in the form x + iy, $x, y \in R$

13. If $a=\cos\theta+i\sin\theta$, then find the value of $\frac{1+a}{1-a}$

14. If $|Z_1|=|Z_2|=1$, then what is the maximum value of $\left|\frac{1}{Z_1}+\frac{1}{Z_2}\right|$?

15. Show that $(Z+1)(\bar{Z}+1)=|Z+1|^2$

16. If $|Z|=2$, show that $\frac{Z-2}{Z+2}$ is purely imaginary.

17. Prove that $\sqrt{i}+\sqrt{-i}=\sqrt{2}$

18. If $|2Z+1|=|Z+2|$, find $|Z|$

19. Show that amp $(1+3i)+amp(3+i)=\frac{\pi}{2}$

20. Find the value of $(-i)^{1/3}$

21. Convert the following complex numbers in the polar form

(i) 2 (ii) $-1-i$

(iii) $-4i$ (iv) $1+i\sqrt{3}$

22. Solve the following equations

(i) $3x^2-4x+\frac{20}{3}=0$

(ii) $\sqrt{5}x^2+x+\sqrt{5}=0$

(iii) $-x^2+x-2=0$

ANSWERS

1. (i) i (ii) $-29+22i$

(iii) $1+2\sqrt{2}i$ (iv) $\frac{8}{65}+\frac{1}{65}i$

(v) $-\frac{1}{4}-\frac{\sqrt{3}}{4}i$ (vi) $-9+46i$

2. (i) $\frac{\sqrt{5}}{14}-\frac{3}{14}i$ (ii) $\cos\theta-i\sin\theta$

3. $-2i$ 4. $x=2,\ y=-2$

5. $\frac{1}{\sqrt{3}}+i$ 7. $\text{Re}(Z)>3$

8. (i) $\frac{1}{5}$ (ii) $\frac{11}{5}$

9. $i-1$

10. (i) $\frac{1}{\sqrt{2}},-\frac{\pi}{4}$ (ii) $4,\frac{2\pi}{3}$

(iii) $\sqrt{2}$, $3\pi/4$ (iv) $3,\pi$

11. 0

12. $-\sqrt{3}+i$

13. $i\cot\frac{\theta}{2}$

18. $|z|=1$

20. i

21. (i) $2(\cos 0 + i\sin 0)$

(ii) $\sqrt{2}\left\{\cos\left(-3\pi/4\right) + i\sin\left(-3\pi/4\right)\right\}$

(iii) $4\left\{\cos\left(-\pi/2\right) + i\sin\left(-\pi/2\right)\right\}$

(iv) $2\left(\cos\pi/3 + i\sin\frac{\pi}{3}\right)$

22. (i) $\frac{2 \pm 4i}{3}$

(ii) $\frac{-1 \pm \sqrt{19}i}{2\sqrt{5}}$

(iii) $\frac{-1 \pm \sqrt{7}i}{2}$

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