MATHEMATICS
CLASS XI
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Binomial Theorem

(Chapter 8)

CONTENTS

CHAPTER
8

BINOMIAL THEOREM

Binomial Theorem for positive integral index :

If $n \in N$ and a, b $\in R$ then

(a + b)n = $a^n + {}^nc_1a^{n-1}b + {}^nc_2a^{n-2}b^2 + {}^nc_3a^{n-3}b^3 + \ldots\ldots\ldots\ldots + {}^nc_r a^{n-r}b^r + \ldots\ldots\ldots + b^n$

Proof : Let n = 1

Then $(a+b)^1 = a+b = {}^1c_0\ a^1 + {}^1c_1\ b^1$

Let n = 2

Then $(a+b)^2 = a^2 + 2ab + b^2 = {}^2c_0\ a^2 + {}^2c_1\ a^{2-1}b^1 + {}^2c_2\ b^2$

$\therefore$ The theorem is true for n = 1, 2.

Let the theorem be true for n = k

Then $(a+b)^k = {}^kc_0\ a^k + {}^kc_1\ a^{k-1}b + {}^kc_2\ a^{k-2}b^2 + {}^kc_3\ a^{k-3}b^3 + \ldots\ldots + {}^kc_{r-1}\ a^{k-(r-1)}b^{r-1} + \ldots\ldots\ldots\ldots\ldots$

$+ \ldots\ldots\ldots\ldots\ldots + {}^kc_r\ a^{k-4}b^r + {}^kc_{r+1}\ a^{k-(r+1)}b^{r+1} + \ldots\ldots + {}^kc_k\ b^k$

Now, $(a+b)^k = (a+b)\,(a+b)^k$

$= (a+b) = \left({}^kc_0\ a^k + {}^kc_1\ a^{k-1}b + {}^kc_2\ a^{k-2}b^2 + {}^kc_3\ a^{k-3}b^3 + \ldots\ldots + {}^kc_{r-1}\ a^{k-(r-1)}b^{r-1}\right.$

$\left. + {}^kc_r\ a^{k-4}b^r + {}^kc_{r+1}\ a^{k-(r+1)}b^{r+1} + \ldots\ldots + {}^kc_k\ b^k\right)$

$= \left({}^kc_0\ a^{k+1} + {}^kc_1\ a^k b + {}^kc_2\ a^{k-1}b^2 + {}^kc_3\ a^{k-2}b^3 + \ldots\ldots + {}^kc_{r-1}\ a^{k-(r-1)}b^{r-1} + {}^kc_r\ a^{k-r+1}b^r\right.$

$\left.{}^kc_{r+1}\ a^{k-r}b^{r+1} + \ldots\ldots + {}^kc_k\ ab^k\right)\left({}^kc_0\ a^k.b + {}^kc_1\ a^{k-1}b^2 + {}^kc_2\ a^{k-2}b^3 + {}^kc_3\ a^{k-3}b^4 + \ldots\ldots + {}^kc_{r-1}\ a^{k-r+1}b^r\right.$

$\left. + {}^kc_r\ a^{k-4}b^{r+1} + {}^kc_{r+1}\ a^{k-r-1}b^{r+2} + \ldots\ldots + {}^kc_k\ b^{k+1}\right)$

$= {}^{k+1}c_0\ a^{k+1} + \left({}^kc_1 + {}^kc_0\right)a^k.b + \left({}^kc_2 + {}^kc_1\right)a^{k-1}b^2 + \left({}^kc_3 + {}^kc_2\right)a^{k-2}b^3 + \ldots\ldots + \left({}^kc_r + {}^kc_{r-1}\right)a^{k+1-r}b^r$

$+ \ldots\ldots\ldots\ldots + {}^{k+1}c_{k+1}\ b^{k+1}$

$= {}^{k+1}c_0\ a^{k+1} + {}^{k+1}c_1\ a^k.b + {}^{k+1}c_2\ a^{k-1}b^2 + {}^{k+1}c_3\ a^{k-2}b^3 + \ldots\ldots\ldots + {}^{k+1}c_r\ a^{k+1-r}b^r + \ldots\ldots\ldots + {}^{k+1}c_{k+1}\ b^{k+1}$

$= {}^{k+1}c_0\ a^{k+1} + {}^{k+1}c_1\ a^{k+1-1}b + {}^{k+1}c_2\ a^{k+1-2}b^2 + {}^{k+1}c_3\ a^{k+1-3}b^3$
$+ \ldots\ldots\ldots + {}^{k+1}c_r\ a^{k+1-r}b^r + \ldots\ldots\ldots + {}^{k+1}c_{k+1}\ a^{k+1}$

$$\begin{bmatrix} {}^kc_0 = a^{k+1} + {}^{k+1}c_0 = 1 \\ {}^kc_k = 1 = {}^{k+1}c_{k+1} = 1 \\ {}^nc_r = 1 = {}^nc_{r-1} = {}^{n+1}c_r \end{bmatrix}$$

$\therefore$ The theorem is true for n = k + 1.

Hence, by principle of mathematical induction the theorem is true for all n.

i.e. $(a+b)^n = {}^nc_0\, a^n + {}^nc_1\, a^{n-1}b + {}^nc_2\, a^{n-2}b^2 + \dots\dots\dots + {}^nc_r\, a^{n-r}b^r + \dots\dots. + {}^nc_n\, b^n$

Note :

(1) The number of terms in the expansion of $(a + b)^n$ is n + 1

(2) $(r + 1)^{th}$ term is ${}^nc_r\, a^{n-r}\, b^r$, it is called the general term which is denoted by T_{r+1}.

Thus, $T_{r+1} = {}^nc_r.a^{n-r}.b^r$

(3) If we replace b by –b in(1) we have

$(a - b)^n = a^n - {}^nc_1a^{n-1}b + {}^nc_2a^{n-2}b^2 - {}^nc_3a^{n-3}b^3 + \dots\dots\dots\dots + (-1)^r\, {}^nc_ra^{n-r}b^r + \dots\dots\dots + (-1)^n b^n$ .... (2)

(4) Replacing a by 1 in (1) & (2) we have

$(1 + b)^n = 1 + {}^nc_1b + {}^nc_2b^2 + {}^nc_3b^3 + \dots\dots\dots\dots + {}^nc_rb^r + \dots\dots\dots + b^n$

$(1 - b)^n = 1 - {}^nc_1b + {}^nc_2b^2 - {}^nc_3b^3 + \dots\dots\dots\dots + (-1)^r\, {}^nc_rb^r + \dots\dots\dots + (-1)^n b^n$

(5) Greatest value of ${}^nC_n$ is ${}^nC_{n/2}$ if n is even and ${}^nC_{\frac{n-1}{2}}$ or ${}^nC_{\frac{n+1}{2}}$ if n is odd.

Example 1 : Find the 5^{th} term in the expansion of $\left(x + \frac{1}{2x}\right)^{10}$

Solution : Here, a = x, $b = \frac{1}{2x}$, n = 10

Now, $T_5 = T_{4+1} = {}^nc_ra^{n-4}.b^r$

$= {}^{10}c_4(x)^{10-4}\left(\frac{1}{2x}\right)^4$ [r = 4]

$= \frac{10\times9\times8\times7\,\lfloor 6}{4\times3\times2\,\lfloor 6}x^6\,\frac{1}{2^4.x^4}$

$= \frac{210}{16}.x^2 = \frac{105}{8}x^2$

Example 2 : Expand $\left(x^2 + \frac{3}{x}\right)^4, x \neq 0$ using Binomial Theorem

Solution : $\left(x^2 + \frac{3}{x}\right)^4$, $a = x^2$ $b = \frac{3}{x}$, $n = 4$

$= \left(x^2\right)^4 + {}^4c_1\left(x^2\right)^3.\left(\frac{3}{x}\right) + {}^4c_2\left(x^2\right)^2.\left(\frac{3}{x}\right)^2 + {}^4c_3\left(x^2\right)^2.\left(\frac{3}{x}\right)^3 + \left(\frac{3}{x}\right)^4$

$= x^8 + 4\times3x^5 + 6\times9x^2 + 4\times27.\frac{1}{x} + \frac{81}{x^4}$

$= x^8 + 12x^5 + 54x^2 + \frac{108}{x} + \frac{81}{x^4}$

Example 3 : Compute 98^5 using Binomial Theorem.

Solution : $98^5 = (100-2)^5$

$= 100^5 - {}^5c_1(100)^4.2 + {}^5c_2(100)^3.2^2 - {}^5c_3(100)^2.2^3 + {}^5c_4 100 \times 2^4 - 2^5$

$= 10000000000 - 5 \times 10000000 \times 2 + 10 \times 1000000 \times 4 - 10 \times 10000 \times 8 + 5 \times 100 \times 16 - 32$

$= 9039207968.$

Example 4 : Which is larger $(1.01)^{10,00,000}$ or 10,000?

Solution : $(1.01)^{10,00,000}$

$= (1+.01)^{1000000}$

$= 1 + {}^{1000000}c_1(.01)^1 + \ldots\ldots\ldots\ldots$

$= 1 + 1000000 \times \frac{1}{100} + \ldots\ldots\ldots\ldots$

$=$ 1 + 10000 + ...

$=$ 10,001 + ... > 10,000

Example 5 : $9^{n+1} - 8n - 9$ is divisible by 64 whenever n is positive integer.

Solution : We have,

$9^{n+1} = (1+8)^{n+1}$

$= 1 + {}^{n+1}c_1 8 + {}^{n+1}c_2 8^2 + {}^{n+1}c_3 8^3 + \ldots\ldots\ldots\ldots + 8^{n+1}$

$= 1 + (n+1).8 + 8^2 + \left({}^{n+1}c_2 + {}^{n+1}c_3.8 + \ldots\ldots\ldots\ldots + 8^{n-1}\right)$

$\Rightarrow 9^{n+1} - 8n - 9 = 64\left({}^{n+1}c_2 + {}^{n+1}c_3.8 + \ldots\ldots\ldots\ldots + 8^{n-1}\right)$

R.H.S is divisible by 64

$\therefore$ L.H.S i.e. $9^{n+1} - 8n - 9$ is divisible by 64

Example 6: Using Binomial theorem prove that $6^n - 5n$ always leaves remainder 1 when divided by 25

Solution : We have,

$6^n = (1+5)^n$

$= 1 + {}^nc_1.5 + {}^nc_2.5^2 + {}^nc_3.5^3 + \ldots\ldots\ldots\ldots + 5^n$

$= 1 + 5n + 5^2\left({}^nc_2 + {}^nc_3.5 + \ldots\ldots\ldots\ldots + 5^{n-2}\right)$

$\Rightarrow 6^n - 5n = 25k + 1$ where $k = {}^nc_2 + {}^nc_3.5 + \ldots\ldots\ldots\ldots + 5^{n-2}$

which show that when $6^n - 5n$ is divided by 25, 1 is the remainder.

Example7 : Prove that

$$\sum_{r=0}^{n} 3^r \, {}^nc_r = 4^n$$

Solution : $\sum_{r=0}^{n} 3^r \, {}^nc_r$

$= \quad {}^nc_0 + {}^nc_1.3 + {}^nc_2.3^2 + {}^nc_3.3^3 + ... + {}^nc_n.3^n$

$= \quad 1 + {}^nc_1.3 + {}^nc_2.3^2 + {}^nc_3.3^3 + ... + 3^n$

$= \quad (1+3)^n = 4^n$

Example 8: Find the coefficient of x^5 in $(x+3)^8$

Solution : Let $(r+1)^{th}$ term contains x^5

$T_{r+1} = {}^nc_r a^{n-r} b^r$ $\qquad$ [$n = 8$, $a = x$, b = 3]

$= {}^8c_r x^{8-r}.3^r$

$\because$ This term contains x^5 therefore

$x^{8-r} = x^5$

$\Rightarrow$ 8 – r = 5

$\Rightarrow$ r = 3

$\therefore$ $(3+1)^{th} = 4^{th}$ term contains x^5

$T_4 = T_{3+1} = {}^8c_3 x^5 3^3$

$= \dfrac{\lfloor 8}{\lfloor 3 \lfloor 5} \times 3^3 \times x^5$

$= \dfrac{8 \times 7 \times 6}{3 \times 2} \times 27 \times x^5$

$= 1512x^5$

$\therefore$ Coefficient of x^5 is 1512

Example 9 : Find the term independent of x in the expansion of $\left(\dfrac{3}{2}x^2 - \dfrac{1}{3x}\right)^6$

Solution : $\left(\dfrac{3}{2}x^2 - \dfrac{1}{3x}\right)^6$

Let, $(r+1)^{th}$ term be independent of x

$T_{r+1} = {}^nc_r a^{n-r} b^r$

$= {}^6c_r \left(\dfrac{3}{2}x^2\right)^{6-r} \left(-\dfrac{1}{3x}\right)^r$

$= {}^6c_r \dfrac{3^{6-r}}{2^{6-r}} x^{12-2r} \left(-\dfrac{1}{3}\right)^r . \dfrac{1}{x^r}$

$= {}^6c_r . \dfrac{3^{6-r}}{2^{6-r}} . \left(-\dfrac{1}{3}\right)^r . x^{12-2r-r}$

But, this term is independent of x

$\therefore \quad x^{12-3r} = x^0$

$\Rightarrow \quad 12 - 3r = 0$

$\Rightarrow \quad r = 4$

$\therefore \quad (4+1)^{th} = 5^{th}$ term is independent of x

$$T_5 = T_{4+1} = {}^nc_4 . \frac{3^2}{2^2}\left(-\frac{1}{3}\right)^4$$

$$= 15 \times \frac{9}{4} \times \frac{1}{81}$$

$$= \frac{5}{12}$$

Example 10 : Find the 13th term in the expansion of $\left(9x - \frac{1}{3\sqrt{x}}\right)^{18}, x \neq 0$

Solution : $T_{13} = T_{12+1} = {}^nc_r \, a^{n-r} b^r$

$$= {}^{18}c_{12} (9x)^{18-12} . \left(-\frac{1}{3\sqrt{x}}\right)^{12} \qquad \left[n = 18, a = 9x, b = -\frac{1}{3\sqrt{x}}, r = 12\right]$$

$$= \frac{18 \times 17 \times 16 \times 15 \times 14 \times 13}{6 \times 5 \times 4 \times 3 \times 2 \times 1} \times 9^6 x^6 \times \frac{1}{3^{12}} \times \frac{1}{x^6}$$

$$= 37128$$

Middle term (s) in the expansion of (a + b)ⁿ

Case-I : When n is even

If n is even there is only one middle term and the term is $\left(\frac{n}{2}+1\right)$th term.

$$T_{\frac{n}{2}+1} = {}^nc_{\frac{n}{2}} \, a^{n-n/2} \, b^{n/2}$$

$$= \frac{\lfloor n}{\lfloor \frac{n}{2} \, \lfloor \frac{n}{2}} a^{n-n/2} \, b^{n/2} = \frac{\lfloor n}{\left(\lfloor n/2\right)^2} a^{n/2} \, b^{n/2}$$

Case-II : When n is odd

If n is odd there are two middle terms. The terms are $\left(\frac{n-1}{2}+1\right)^{th}$ and $\left(\frac{n+1}{2}+1\right)^{th}$ terms.

$$T_{\frac{n-1}{2}+1} = {}^nc_{\frac{n-1}{2}} \, a^{n-\frac{n-1}{2}} \, b^{\frac{n-1}{2}}$$

$$= \frac{\lfloor n}{\lfloor \frac{n-1}{2} \lfloor \frac{n+1}{2}} a^{\frac{n+1}{2}} b^{\frac{n-1}{2}}$$

$$T_{\frac{n-1}{2}+1} = {}^{n}c_{\frac{n+1}{2}} a^{n-\frac{n+1}{2}} b^{\frac{n+1}{2}}$$

$$= \frac{\lfloor n}{\lfloor \frac{n+1}{2} \lfloor \frac{n-1}{2}} a^{\frac{n-1}{2}} b^{\frac{n+1}{2}}$$

Example 11: Find the middle terms in the expansion of

(i) $\left(3-\frac{x^3}{6}\right)^7$

Solution : $\left(3-\frac{x^3}{6}\right)^7$

Here n = 7, odd number

∴ There are two middle terms

The terms are $\left(\frac{7-1}{2}+1\right)^{th}$ and $\left(\frac{7+1}{2}+1\right)^{th}$

i.e. 4^{th} and 5^{th} terms

$T_4 = T_{3+1} = {}^{n}c_r\, a^{n-r} b^r$

$= {}^{7}c_3\, (3)^{7-3} \left(\frac{-x^3}{6}\right)^3$

$= 35 \times 3^4\, (-1)\, \frac{x^9}{6^3} = \frac{-105}{8} x^9$

(ii) $\left(\frac{x}{3}+9y\right)^{10}$

Solution : Here n = 10, even number

There are only one middle terms

∴ The terms is $\left(\frac{10}{2}+1\right)^{th}$ i.e. 6^{th} term

$T_6 = T_{5+1} = {}^{n}c_r\, a^{n-r} b^r$

$= {}^{10}c_5 \left(\frac{x}{3}\right)^5 (9y)^5$

$$= \frac{10\times9\times8\times7\times6}{5\times4\times3\times2\times1}\times\frac{x^5}{3^5}\times9^5.y^5 = 252\times3^5\times x^5y^5$$

Example 12 : Show that the middle term in the expansion of $(1+x)^{2n}$ is $\frac{1.3.5....(2n-1)}{\lfloor n}.2^n x^n$

Solution : $(1+x)^{2n}$

Here n = 2n, even number
There is only one middle term

$\therefore$ The terms is $\left(\frac{2n}{2}+1\right)^{th}$ i.e. $(1+n)^{2n}$ term

$$T_{n+1} = {}^nc_r\, a^{n-r}b^r$$

$$= {}^{2n}c_n\, 1^{2n-n}\, x^n$$

$$= \frac{\lfloor 2n}{\lfloor n\ \lfloor n}\times x^n$$

$$= \frac{2n(2n-1)\,(2n-2)\,(2n-3).......3.2.1}{\lfloor n\ \lfloor n}\,x^n$$

$$= \frac{\{2n(2n-2)\,(2n-4)\,.......4.2\}\,\{(2n-1)\,(2n-3)\,.......3.1\}}{\lfloor n\ \lfloor n}\,x^n$$

$$= \frac{2^n\,\{n(n-1)\,(n-2)\,.......2.1\}\,\{(2n-1)\,(2n-3)\,.......3.1\}}{\lfloor n\ \lfloor n}\,x^n$$

$$= \frac{1.3.......(2n-1)}{\lfloor n}2^n\,x^n$$

Binomial Coefficient –

We have,

$(a + x)^n = {}^nc_0\, a^n + {}^nc_1\, a^{n-1}x + {}^nc_2\, a^{n-2}x^2 + {}^nc_r\, a^{n-r}x^r + ...{}^nc_r\, x^n$

${}^nc_0\, , {}^nc_1\, , {}^nc_2\, ,.........{}^nc_r\, ,........, {}^nc_n$ are called binomial coefficients. In short we write c_0 for ${}^nc_0$, $c_1$for ${}^nc_1$, ...c_n for ${}^nc_n$

Example13 : If $c_0\, , c_1\, , c_2\, ,........., c_n$ are coefficients in the expansion of $(1 + x)^n$, then show that

(i) $c_0 + c_1 + c_2 + ... + c_n = 2^n$

(ii) $c_0 + c_2 + c_4 + ... = c_1 + c_3 + c_5 + ... = 2^{n-1}$

Solution : We have

$(1+x)^n = c_0 + c_1x + c_2x^2 + c_3x^3 + c_4x^4 + ... + c_nx_n$ .... (1)

Putting x = 1, we have

$2^n = c_0 + c_1 + c_2 + c_3 + c_4 + c_5 + c_6 + ... + c_n$

$\Rightarrow \quad c_0 + c_1 + c_2 + c_3 + c_4 + c_5 + c_6 + ... + c_n = 2$ .... (2)

Putting x = –1 in (1) we have

$0 = c_0 - c_1 + c_2 - c_3 + c_4 - c_5 + c_6 + ... + (-1)^n c_n$

$\Rightarrow \quad c_0 - c_1 + c_2 - c_3 + c_4 - c_5 + c_6 + ... = 0$ .... (3)

(2) + (3) $\Rightarrow 2(c_0 + c_2 + c_4 + c_6 + ...) = 2^n$

$\Rightarrow c_0 + c_2 + c_4 + c_6 + ... = \frac{2^n}{2} = 2^{n-1}$ .... (4)

(2) – (3) $\Rightarrow 2(c_1 + c_3 + c_5 + ...) = 2^n$

$\Rightarrow c_1 + c_3 + c_5 + ... = \frac{2^n}{2} = 2^{n-1}$ .... (5)

From (4) and (5)

$\Rightarrow c_0 + c_2 + c_4 + c_6 + ... = c_1 + c_3 + c_5 + ... = 2^{n-1}$

Example 14: Find a if the 17^{th} and 18^{th} terms in the expansion of $(2 + a)^{50}$ are equal.

Solution : $T_{17} = T_{16+1} = {}^n c_r\, a^{n-r} b^r$

$= {}^{50}c_{16}\, (2)^{50-16} a^{16}$

$= {}^{50}c_{16} \times 2^{34} \times a^{16}$

$T_{18} = T_{17+1} = {}^{50}c_{17}\, (2)^{50-17} a^{17}$

$= {}^{50}c_{17} \times 2^{33} \times a^{17}$

A/Q, ${}^{50}c_{16} \times 2^{34} \times a^{16} = {}^{50}c_{17} \times 2^{33} \times a^{17}$

$\Rightarrow \frac{\lfloor 50}{\lfloor 16 \lfloor 34} \times 2 = \frac{\lfloor 50}{\lfloor 17 \lfloor 33} \times a$

$\Rightarrow \frac{1}{\lfloor 16\ 34 \lfloor 33} \times 2 = \frac{1}{17 \lfloor 16\ \lfloor 33} \times a$

$\Rightarrow$ a = 1

Example 15: The 2nd, 3rd and 4th terms in the binomial expansion $(x + a)^n$ are 240, 720, and 1080. Find x, a and n.

Solution : $T_2 = T_{1+1} = {}^n c_1\, x^{n-1} a$

$T_3 = T_{2+1} = {}^n c_2\, x^{n-2} a^2$

$T_4 = T_{3+1} = {}^n c_3\, x^{n-3} a^3$

A/Q, ${}^n c_1\, x^{n-1} a = 240$ .... (1)

${}^n c_2\, x^{n-2} a^2 = 720$ .... (2)

${}^n c_3\, x^{n-3} a^3 = 1080$ .... (3)

(2) ÷ (1) $\Rightarrow \dfrac{{}^nc_2\, x^{n-2}a^2}{{}^nc_1\, x^{n-1}a} = \dfrac{720}{240}$

$\Rightarrow \dfrac{\dfrac{\lfloor n}{\lfloor 2\ \lfloor n-2}.\, a}{\dfrac{\lfloor n}{\lfloor 1\ \lfloor n-1}.\, x} = 3$

$\Rightarrow \dfrac{\lfloor n-1}{2\ \lfloor n-2}\, a = 3x$

$\Rightarrow \dfrac{(n-1)\lfloor n-2}{2\ \lfloor n-2}\, a = 3x$

$\Rightarrow (n-1)\, a = 6x$ .... (4)

(3) ÷ (2) $\Rightarrow \dfrac{{}^nc_3\, x^{n-3}a^3}{{}^nc_2\, x^{n-2}a^2} = \dfrac{1080}{720}$

$\Rightarrow \dfrac{\dfrac{\lfloor n}{\lfloor 3\ \lfloor n-3}.\, a}{\dfrac{\lfloor n}{\lfloor 1\ \lfloor n-2}.\, x} = \dfrac{1}{2}$

$\Rightarrow \dfrac{\lfloor 2\ \lfloor n-2.\, a}{\lfloor 3\ \lfloor n-3.\, x} = \dfrac{3}{2}$

$\Rightarrow \dfrac{\lfloor 2\ (n-2)\lfloor n-2}{3\lfloor 2\ \lfloor n-3}\dfrac{a}{x} = \dfrac{3}{2}$

$\Rightarrow (n-2)\, a = \dfrac{9}{2}x$ .... (5)

(5) ÷ (4) $\Rightarrow \dfrac{n-2}{n-1} = \dfrac{9/2}{6}$

$\Rightarrow \dfrac{n-2}{n-1} = \dfrac{9}{12} = \dfrac{3}{4}$

$\Rightarrow$ 3n – 3 = 4n – 8

$\Rightarrow$ n = 5

(4) $\Rightarrow$ 4a = 6x

$\Rightarrow$ 2a = 3x ...... (4)′

(5) $\Rightarrow 3a = \dfrac{9}{2}x$

$\Rightarrow a = \dfrac{3}{2}x$

(1) $\Rightarrow\ ^5c_1\, x^{5-1}a = 240$

$\Rightarrow\ 5\times x^4.\dfrac{3x}{2} = 240$

$\Rightarrow\ x^5 = 32 = 2^5$

$\Rightarrow$ x = 2

(4)′ $\Rightarrow$ 2a = 3.2

$\Rightarrow$ a = 3

Example 16: If the coefficients a^{r-1}, a^r, a^{r+1} in the expansion of $(1+a)^n$ are in arithmetic progression, then show that $n^2 - n(4r+1) + 4r^2 - 2 = 0$

Solution : We have

$(1+a)^n\ = {}^nc_1\, a + {}^nc_2\, a^2 + {}^nc_3\, a^3 + ... + {}^nc_{r-1}\, a^{r-1} + {}^nc_r\, a^r + {}^nc_{r+1}\, a^{r+1} + ... + a^n$

Coefficient of a^{r-1} is ${}^nc_{r-1}$

Coefficient of a^r is ${}^nc_r$

Coefficient of a^{r+1} is ${}^nc_{r+1}$

Since, the coefficient of arithmetic progression

$\therefore\ {}^nc_r - {}^nc_{r-1} = {}^nc_{r+1} - {}^nc_r$

$\Rightarrow\ {}^nc_{r-1} + {}^nc_{r+1} = 2.{}^nc_r$

$$\Rightarrow\ \frac{\underline{|n}}{\underline{|r-1}\ \underline{|n-r+1}} + \frac{\underline{|n}}{\underline{|r+1}\ \underline{|n-r-1}} = 2.\frac{\underline{|n}}{\underline{|r}\ \underline{|n-r}}$$

$$\Rightarrow\ \frac{1}{\underline{|r-1}\ (n-r+1)(n-r)\underline{|n-r-1}} + \frac{1}{(r+1)\,r\ \underline{|r-1}\underline{|n-r-1}} = \frac{2}{r\ \underline{|r-1}(n-r)\underline{|n-r-1}}$$

$$\Rightarrow\ \frac{1}{(n-r+1)(n-r)} + \frac{1}{(r+1)r} = \frac{2}{r\,(n-r)}$$

$$\Rightarrow\ \frac{r(r+1)+(n-r)(n-r+1)}{r\,(r+1)(n-r)(n-r+1)} = \frac{2}{r\,(n-r)}$$

$\Rightarrow\ r(r+1) + (n-r)\,(n-r+1) = 2(r+1)\,(n-r+1)$

$\Rightarrow\ r^2 + r + n^2 - nr + n - nr + r^2 - r = 2nr - 2r^2 + 2r + 2n - 2r + 2$

$\Rightarrow\ n^2 - 4rn - n + 4r^2 - 2 = 0$

$\Rightarrow\ n^2 - n(4r+1) + 4r^2 - 2 = 0$

Example17 : Find the coefficient of x^5 in the product $(1 + 2x)^6 (1 - x)^7$ using binomial theorem.

Solution : $(1 + 2x)^6 (1 - x)^7$

$$= \left\{1 + {}^6c_1\times1\times2x + {}^6c_2\times1\times(2x)^2 + {}^6c_3\times1\times(2x)^3 + {}^6c_4\times1\times(2x)^4 + {}^6c_5\times1\times(2x)^5 + (2x)^6\right\}$$

$$\left\{1 - {}^7c_1x + {}^7c_2x^2 - {}^7c_3x^3 + {}^7c_4x^4 - {}^7c_5x^5 + {}^7c_6x^6 - {}^7c_7x^7\right\}$$

Coefficient of x^5 is = $-{}^7c_5\left({}^7c_4\times{}^6c_1\times 2\right)-\left({}^6c_2\times 2^2\times{}^7c_3\right)+\left({}^6c_3\times 2^3\times{}^7c_2\right)-\left({}^6c_4\times 2^4\times{}^7c_1\right)+{}^6c_5\times 2^5$

$= -21(35\times 6\times 2)-(15\times 4\times 35)+(20\times 8\times 21)-(15\times 16\times 7)+6\times 32$

= – 21 + 420 – 2100 + 3360 – 1680 + 192

= 3972 – 3801 = 171

Example 18 : Show that the coefficient of the middle term in the expansion of $(1 + x)^{2n}$ is equal to the xsum of the coefficient of two middle terms in the expansion of $(1 + x)^{2n-1}$

Solution : $(1 + x)^{2n}$

2n is even number. The middle term is $\left(\frac{2n}{2}+1\right)^{th}$ term = $(1 + x)^{th}$ term

$T_{n+1} = {}^{2n}c_n\, x^n$

Coefficient is = ${}^{2n}c_n$

$(1 + x)^{2n}$

2n – 1 is odd no. The two-middle terms.

The terms are $\left(\frac{2n-1-1}{2}+1\right)^{th}$ and $\left(\frac{2n-1+1}{2}+1\right)^{th}$

term i.e. n^{th} and = $(n + 1)^{th}$ terms

$T_n = T_{(n-1)+1} = {}^{2n-1}c_{n-1}\, x^{2n-(n-1)}$

Coefficient = ${}^{2n-1}c_{n-1}$

$\therefore$ $T_{(n-1)} = {}^{2n-1}c_n\, x^{2n-n-1}$, Coefficient = ${}^{2n-1}c_n$

Now, ${}^{2n-1}c_{n-1}+{}^{2n-1}c_n={}^{(2n-1)+1}c_n={}^{2n}c_n$

Example 19 : Find $(x + 1)^6 + (x – 1)^6$. Hence or otherwise evaluate $\left(\sqrt{2}+1\right)^6+\left(\sqrt{2}-1\right)^6$

Solution : $(x + 1)^6 + (x – 1)^6$

$= \left(x^6+{}^6c_1x^5+{}^6c_2x^4+{}^6c_3x^3+{}^6c_4x^2+{}^6c_5x+1\right)+\left(x^6-{}^6c_1x^5+{}^6c_2x^4-{}^6c_3x^3+{}^6c_4x^2-{}^6c_5+1\right)$

$= 2\left(x^6+{}^6c_2x^4+{}^6c_4x^2+1\right)$

$= 2\left(x^6+15x^4+15x^2+1\right)$

Putting $x=\sqrt{2}$ we have

$\left(\sqrt{2}+1\right)^6+\left(\sqrt{2}-1\right)^6 = 2\left\{\left(\sqrt{2}\right)^6+15\left(\sqrt{2}\right)^4+15\left(\sqrt{2}\right)^2+1\right\}$

$= 2\,(8+60+30+1)$

$= 2\times 99=198$

Example 20: If a_1, a_2, a_3, a_4 are the coefficients of any four consecutive terms in the expansion of $(x + 1)^n$, prove that

$$\frac{a_1}{a_1+a_2}+\frac{a_3}{a_3+a_4}=\frac{2a_1}{a_2+a_4}$$

Solution : Let a_1, a_2, a_3, a_4 be the coefficient of the $(r + 1)^{th}$, $(r + 2)^{th}$, $(r + 3)^{th}$ and $(r + 4)^{th}$ terms respectively.

$$T_{r+1} = {}^n c_r \, x^r$$

$$T_{r+2} = T_{(r+1)+1} = {}^n c_{r+1} \, x^{r+1}$$

$$T_{r+3} = T_{(r+2)+1} = {}^n c_{r+2} \, x^{r+2}$$

$$T_{r+4} = T_{(r+3)+1} = {}^n c_{r+3} \, x^{r+3}$$

$$\therefore \; a_1 = a_1 = {}^n c_r \,, a_2 = {}^n c_{r+1} \,, a_3 = {}^n c_{r+2} \,, a_4 = {}^n c_{r+3}$$

$$\frac{a_1}{a_1 + a_2} + \frac{1}{1+\frac{a_2}{a_1}} \qquad \text{........... (1)}$$

Now, $$\frac{a_2}{a_1} = \frac{{}^n c_{r+1}}{{}^n c_r} + \frac{1}{1+\frac{a_2}{a_1}}$$

$$= \frac{\dfrac{\lfloor\underline{n}}{\lfloor\underline{r+1}\;\lfloor\underline{n-r-1}}}{\dfrac{\lfloor\underline{n}}{\lfloor\underline{r}\;\lfloor\underline{n-r}}}$$

$$= \frac{\lfloor\underline{r}\;\lfloor\underline{n-r}}{\lfloor\underline{r+1}\;\lfloor\underline{n-r-1}} = \frac{\lfloor\underline{r}\;(n-r)\lfloor\underline{n-r-1}}{(r+1)\lfloor\underline{r}\;\lfloor\underline{n-r-1}} = \frac{n-r}{r+1}$$

$$(1) \Rightarrow \frac{1}{1+\frac{n-r}{r+1}} = \frac{r+1}{r+1+n-r} = \frac{r+1}{n+1}$$

$$\frac{a_3}{a_3 + a_4} = \frac{1}{1+\frac{a_4}{a_3}}$$

Now, $$\frac{a_4}{a_3} = \frac{{}^n c_{r+3}}{{}^n c_{r+2}}$$

$$= \frac{\dfrac{\lfloor\underline{n}}{\lfloor\underline{r+3}\;\lfloor\underline{n-r-3}}}{\dfrac{\lfloor\underline{n}}{\lfloor\underline{r+2}\;\lfloor\underline{n-r-2}}}$$

$$= \frac{\lfloor\underline{r+2}\;\lfloor\underline{n-r-2}}{\lfloor\underline{r+3}\;\lfloor\underline{n-r-3}}$$

$$= \frac{\lfloor r+2 \;\; (n-r-2)\lfloor n-r-3}{(r+3)\lfloor r+2 \;\; \lfloor n-r-3} = \frac{n-r-2}{r+3}$$

$$\therefore \quad \frac{a_3}{a_3+a_4} = \frac{1}{1+\frac{n-r-2}{r+3}} = \frac{r+3}{r+3+n-r-2} = \frac{r+3}{n+1}$$

Now, $\frac{2a_2}{a_2+a_3} = \frac{2}{1+\frac{a_3}{a_2}}$

$$= \frac{2}{1+\frac{n-r-1}{r+1}}$$

$$= \frac{2(r+2)}{r+2+n-r-1}$$

$$= \frac{2(r+2)}{n+1}$$

$$\frac{a_3}{a_2} = \frac{{}^nc_{r+2}}{{}^nc_{r+1}}$$

$$= \frac{\frac{\lfloor n}{\lfloor r+2 \;\; \lfloor n-r-2}}{\frac{\lfloor n}{\lfloor r+1 \;\; \lfloor n-r-1}}$$

$$= \frac{\lfloor r+1 \;\; \lfloor n-r-1}{\lfloor r+2 \;\; \lfloor n-r-2}$$

$$= \frac{n-r-1}{r+2}$$

Now $\frac{a_1}{a_1+a_2} + \frac{a_3}{a_3+a_4} = \frac{r+1}{n+1} + \frac{r+3}{n+1}$

$$= \frac{r+1+r+3}{n+1}$$

$$= \frac{2(r+2)}{n+1} = \frac{2a_2}{a_2+a_3}$$

Example 21 : Find the coefficient of x^7 in $\left(ax^2+\frac{1}{bx}\right)^{11}$ and x^{-7} in $\left(ax-\frac{1}{bx^2}\right)^{11}$ and find the relation between a and b so that these coefficient are equal.

Solution : $\left(ax^2+\frac{1}{bx}\right)^{11}$

Let, (r + 1)th term contains x^7

$$T_{r+1} = {}^{11}c_r \left(ax^2\right)^{11-r} . \left(\frac{1}{bx}\right)^r$$

$$= {}^{11}c_r \; a^{11-r} \; x^{22-2r} . \frac{1}{b^r} . x^{-r}$$

$$= {}^{11}c_r \; a^{11-r} \; \frac{1}{b^r} . x^{22-2r-r}$$

A/Q $\quad 22 - 3r = 7$

$\Rightarrow$ 3r = 15 $\Rightarrow$ r = 5

$\therefore$ $(5 + 1)^{th} = 6^{th}$ term contains x^7.

Coefficient of x^7 is = ${}^{11}c_5\, a^6 / a^6$

$$\left(ax - \frac{1}{bx^2}\right)^2$$

Let, $(s + 1)^{th}$ term contains x^{-7}

$$T_{s+1} = {}^{11}c_s\, (ax)^{11-s} \left(\frac{1}{bx^2}\right)^s$$

$$= {}^{11}c_s\, (ax)^{11-s} . x^{11-s} \left(-\frac{1}{b}\right)^s . x^{-2s}$$

$$= {}^{11}c_s\, a^{11-s} \left(-\frac{1}{b}\right)^s . x^{11-s-2s}$$

But this term contains x^{-7}

$\therefore$ –7 = 11 – s – 2s

$\Rightarrow$ 3s = 18 $\Rightarrow$ s = 6

$\therefore$ $(6 + 1)^{th} = 7^{th}$ term contains x^{-7}

Coefficient of x^{-7} is = ${}^{11}c_6\, a^{11-6} \left(-\frac{1}{b}\right)^6$

$$= {}^{11}c_6\, a^5 / b^6$$

If the coefficients are equal, then

$${}^{11}c_5\, a^6 / b^5 = {}^{11}c_6\, a^5 / b^6$$

$\Rightarrow$ ab = 1

Example 22 : In the expansion of $(x + a)^n$, if the sum of odd terms be p and sum of even terms be q, prove that

(i) $p^2 - q^2 = (x^2 - a^2)^n$

(ii) $4pq = (x + a)^{2n} - (x - a)^{2n}$

Solution : $(x + a)^n$

$$= x^n + {}^nc_1\, x^{n-1}a + {}^nc_2\, x^{n-2}a^2 + {}^nc_3\, x^{n-3}a^3 + \ldots\ldots\ldots + a^n$$

then

$$p = x^n + {}^n C_2 x^{n-2} a^2 + {}^n C_2 x^{n-2} a^2 + {}^n C_4 x^{x-4} a^4 + \ldots\ldots\ldots.$$

$$q = {}^nc_1\, x^{n-1}a + {}^nc_3\, x^{n-3}a^3 + \ldots\ldots\ldots\ldots$$

(i) Now $p^2 - q^2 = (p + q)(p - q)$

$$= \left(x^n + {}^nc_1\, x^{n-1}a + {}^nc_2\, x^{n-2}a^2 + \ldots\ldots\ldots + a^n\right)\left\{x^n - {}^nc_1\, x^{n-1}a + {}^nc_2\, x^{n-2}a^2 - \ldots\ldots\ldots + (-1)^n a^n\right\}$$

$$= (x+a)^n (x-a)^n = \left\{(x+a)(x-a)\right\}^n = \left(x^2 - a^2\right)^n$$

We have 4pq $= (p+q)^2 - (p-q)^2$

$$= \left\{(x+a)^n\right\}^2 - \left\{(x-a)^n\right\}^2$$

$$= (x+a)^{2n} - (x-a)^{2n}$$

EXERCISE : 8

1. Expand by using binomial theorem

   (i) $(2x+3y)^5$ (ii) $\left(\frac{2x}{3}-\frac{3}{2x}\right)^4$

2. Find the values of the following using binomial theorem

   (i) $(999)^3$ (ii) $(10.1)^4$ (iii) $(1.02)^6$

   correct to five decimal places.

3. Which number is larger?

   $(1.2)^{4000}$ or 800

4. Find $(a+b)^4-(a-b)^4$

   Hence evaluate $\left(\sqrt{3}+\sqrt{2}\right)^4-\left(\sqrt{3}-\sqrt{2}\right)^4$

5. By using binomial theorem prove that $2^{3n}-7n-1$ is divisible by 49, for all n$\in$ N.

6. Which term in the expansion of $\left(x-\frac{2}{x}\right)^{11}$ contains x^{-3}.

7. If the 21st and 22nd terms in the expansion of $(1+x)^{44}$ are equal, then find the value of x.

8. Find the middle term in the expansion of $\left(x-\frac{1}{x}\right)^{10}$

9. If the co-efficients of three consecutive terms in the expansion of $(1+x)^n$ are in the ratio 1 : 7 : 42, find n.

10. The first three terms in the expansion of a binomical are 1, 10 and 40 Find the expansion.

11. Expand $\left(1-x+x^2\right)^4$ using binomial,

12. In the binomical expansion of $\left(3\sqrt{3}+\sqrt{2}\right)^5$ find the term which does not contain irrational expression.

13. Find the co-efficient of x^3 in the expansion of the product $(1+2x)^6 \cdot (1+x)^7$

14. Find the value of k so that the term independent of x in the expansion of $\left(\sqrt{x}+\frac{x}{x^2}\right)^{10}$ is 405

15. If the fourth term in the expansion of $\left(1-\frac{x}{n}\right)^n$ equals $\frac{7}{8}$, when $x=-2$ and n is a positive integer, calculate n

16. Find the 11th term from the end in the expansion of $\left(2x-\frac{1}{x^2}\right)^{25}$

17. If $(1+ax)^n = 1+8x+24x^2+\ldots\ldots\ldots\ldots$, $n\in N$, then find the values of a and n.

18. If the co-efficients of $(r-5)$ th and $(2r-1)$ th terms in the expansion of $(1+x)^{34}$ are equal, find r.

19. In the binomial expansion of $(1+x)^{m+n}$ prove that the co-efficients of x^m and x^n are equal.

20. Find the two consecutive terms in the expansion of $(3+2x)^{74}$, so that the co-efficients of powers of x are euqal

21. Let n be a positive integer. If the co-efficients of 2nd, 3rd, 4th terms in the expansion of $(1+x)^n$ are in A. P then find the value of n.

22. In the expansion of $(1+x)^{50}$, what is the sum of the co-efficients of the odd powers of x.

23. Show that the greatest co-efficient in the expansion of $(1+x)^{2n+2}$ is $\dfrac{(2n+2)!}{\{(n+1)!\}^2}$

24. If x^r occurs in the expansion of $\left(x+\dfrac{1}{x}\right)^n$, then find its co-efficient.

25. In the binomial expansion of $(a-b)^n$, $n \geq 5$ the sum of 5th and 6th terms is zero. show that $\dfrac{a}{b}=\dfrac{n-4}{5}$

26. If the middle term in the expansion of $\left(x^2+\dfrac{1}{x}\right)^n$ is $924x^6$, then find the value of n.

27. If $\dfrac{T_2}{T_3}$ in the expansion of $(a+b)^n$ and $\dfrac{T_3}{T_4}$ in the expansion of $(a+b)^{n+3}$ are equal, then find the value of n.

28. Find the value of $\left(\sqrt{2}+1\right)^6+\left(\sqrt{2}-1\right)^6$

29. Find the terms independent of x in the expansion of $\left(x^2-\dfrac{1}{3x}\right)^9$

30. If the co-efficient of x^7 and x^8 in $\left(2+\dfrac{x}{3}\right)^n$ are equal, then find n.

ANSWERS

1.(i) $32x^5+240x^4y+720x^3y^2+1080x^2y^3+810xy^4+243y^5$

(ii) $\dfrac{16}{81}x^4-\dfrac{16}{9}x^2+6-\dfrac{9}{x^2}+\dfrac{81}{16x^4}$

2.(i) 97002999 (ii) 10406.0401 (iii) 1.12616

3. $(1.2)^{4000} > 800$

4. $8ab\left(a^2+b^2\right), 104\sqrt{6}$

6. 8th 7. $\frac{7}{8}$

8. ${}^{-10}C_5$

9. 55

10. $(1+2)^5$

11. $1-4x+10x^2-10x^3+19x^4-16x^5+10x^6-4x^7+x^8$

12. 3rd term, 60

13. -43

14. k = 3, –3

15. 8

16. $\frac{-\lfloor 25}{\lfloor 15 . \lfloor 10} \frac{2^{10}}{x^{20}}$

17. a = 2 , n= 4

18. 14

21. 7

22. 2^{49}

24. $\frac{n!}{\left(\frac{n+r}{2}\right)!\left(\frac{n-r}{2}\right)1}$

25. n = 12

27. 5

28. 198

29. $\frac{28}{243}$

30. 55

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